Fair Allocations for Smoothed Utilities

Yushi Bai, Uriel Feige, Paul Gölz & Ariel Procaccia

Fair Item Allocation

5	5	0	4
2	1	2	2
2	1	1	3

Setting:
- indivisible goods
- additive utilities

Envy Freeness (EF)

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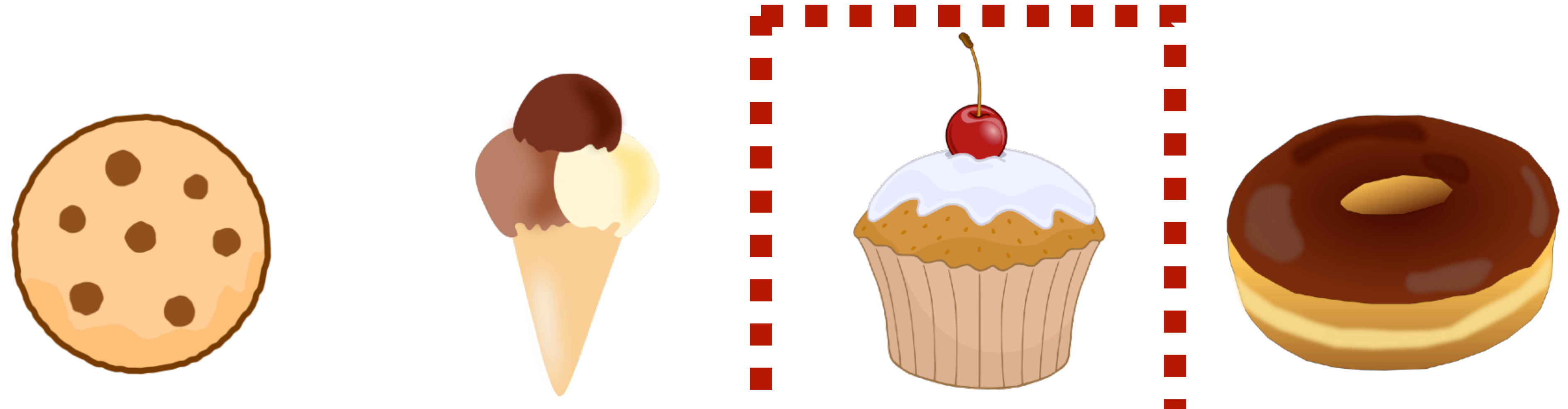

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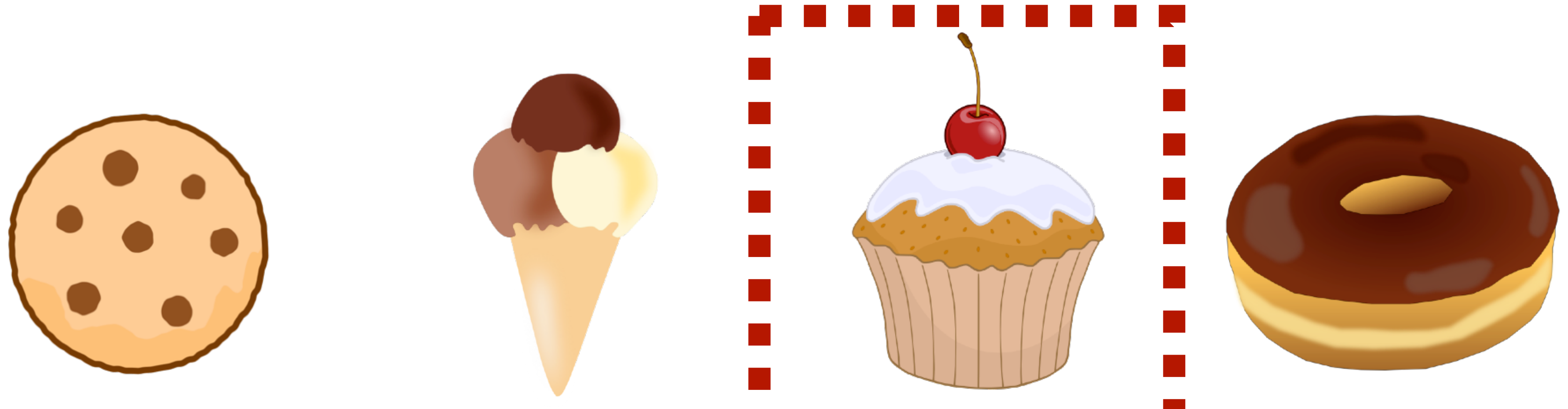

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How to capture this mathematically?

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				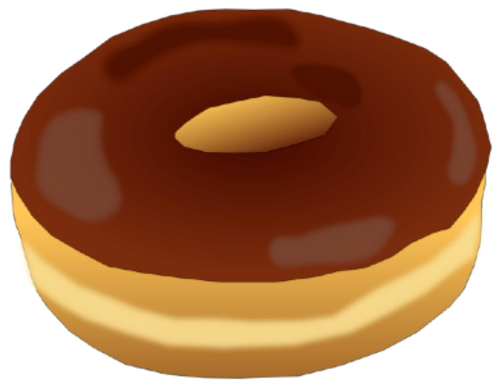
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Theorem [MS21]. If $m \in \Omega(n \log n / \log \log n)^*$, envy-free allocations exist in i.i.d. model with probability $\to 1$.

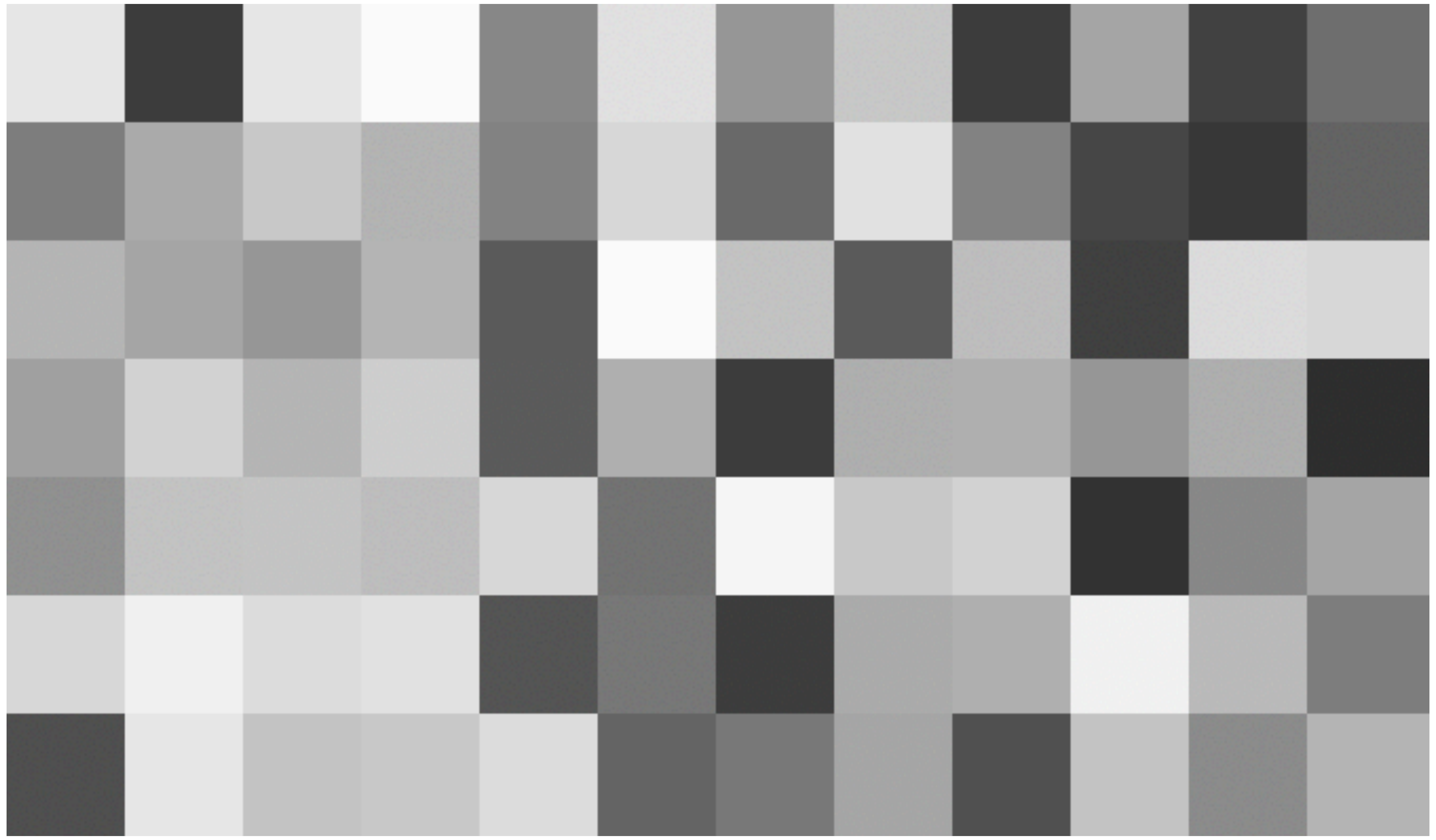
n: number of agents
m: number of items

* under some assumption on $\mathscr{D}$

Manurangsi, P. & Suksompong, W. Closing Gaps in Asymptotic Fair Division. *SIAM J. Discrete Math.* 35, 668–706 (2021).

Limitations of Previous Work

i.i.d. values



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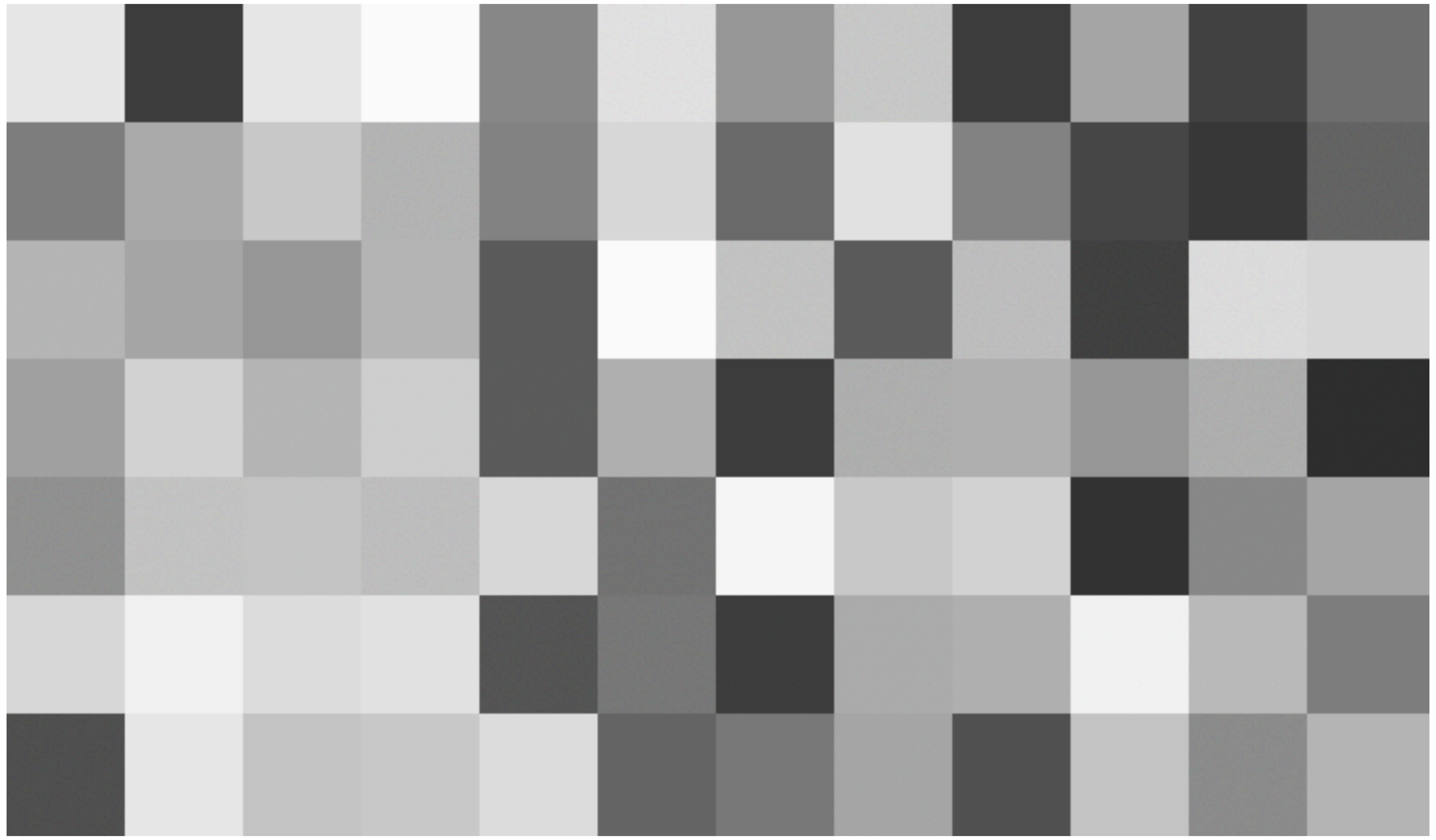
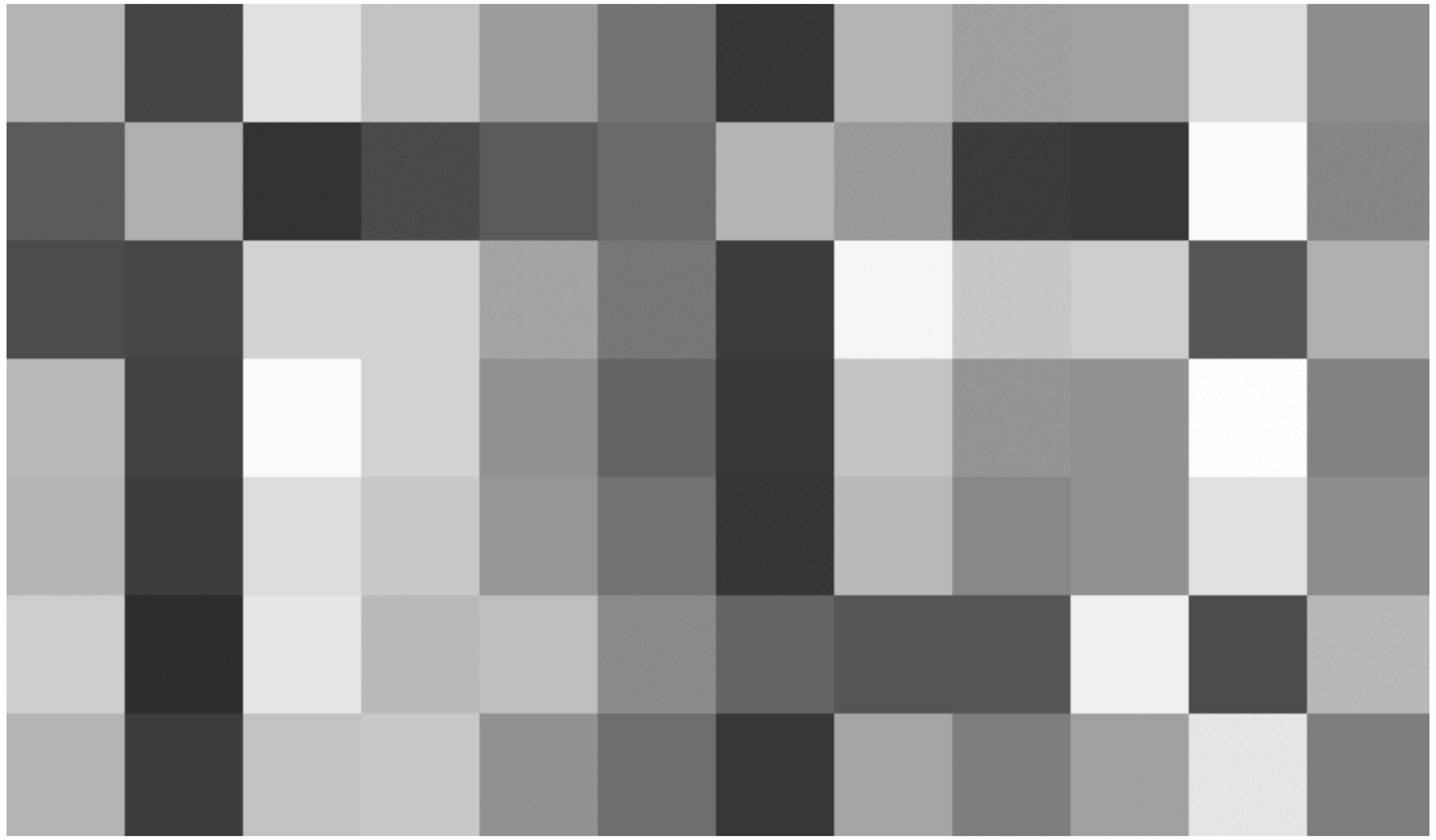


values with structure



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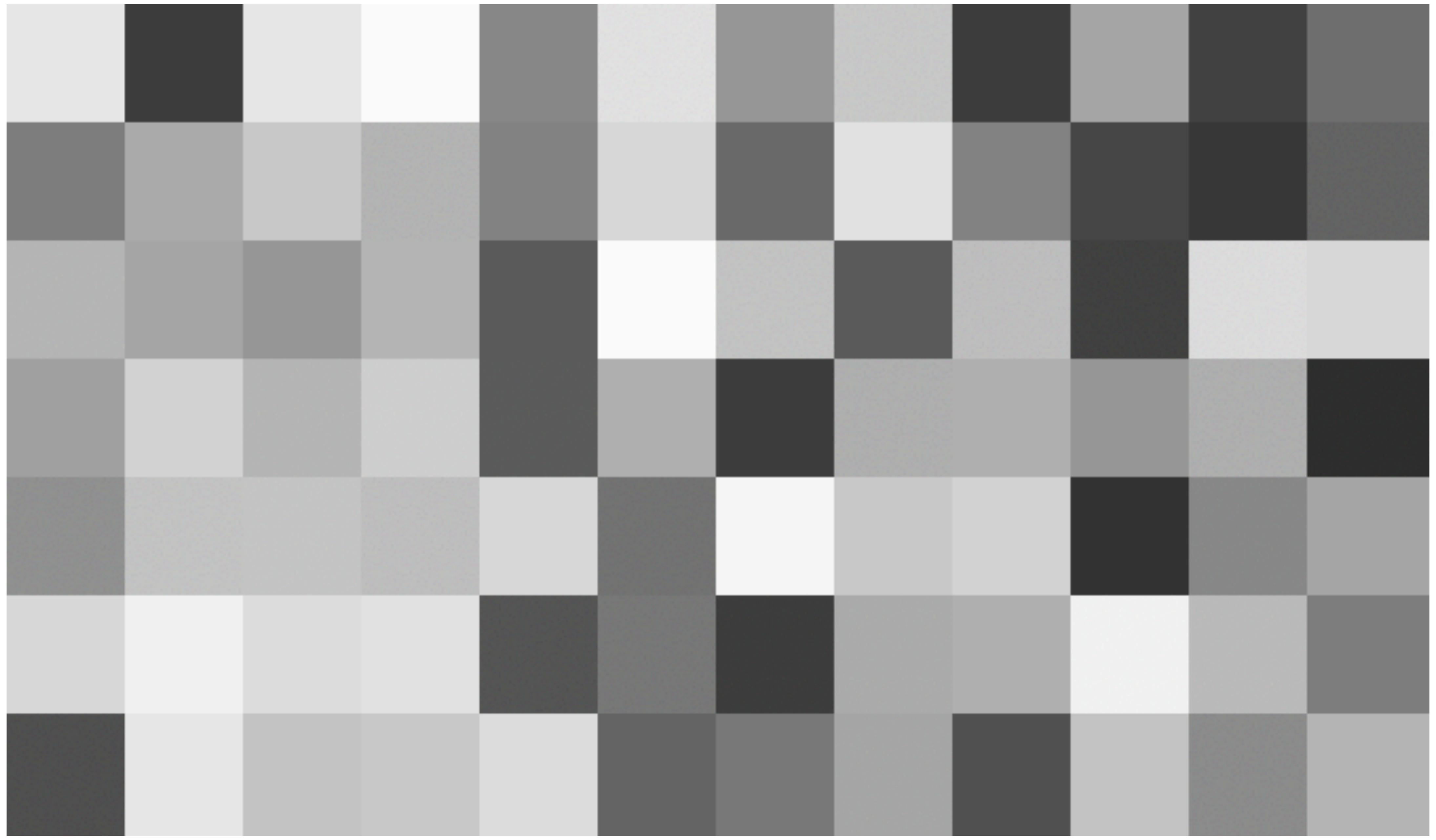

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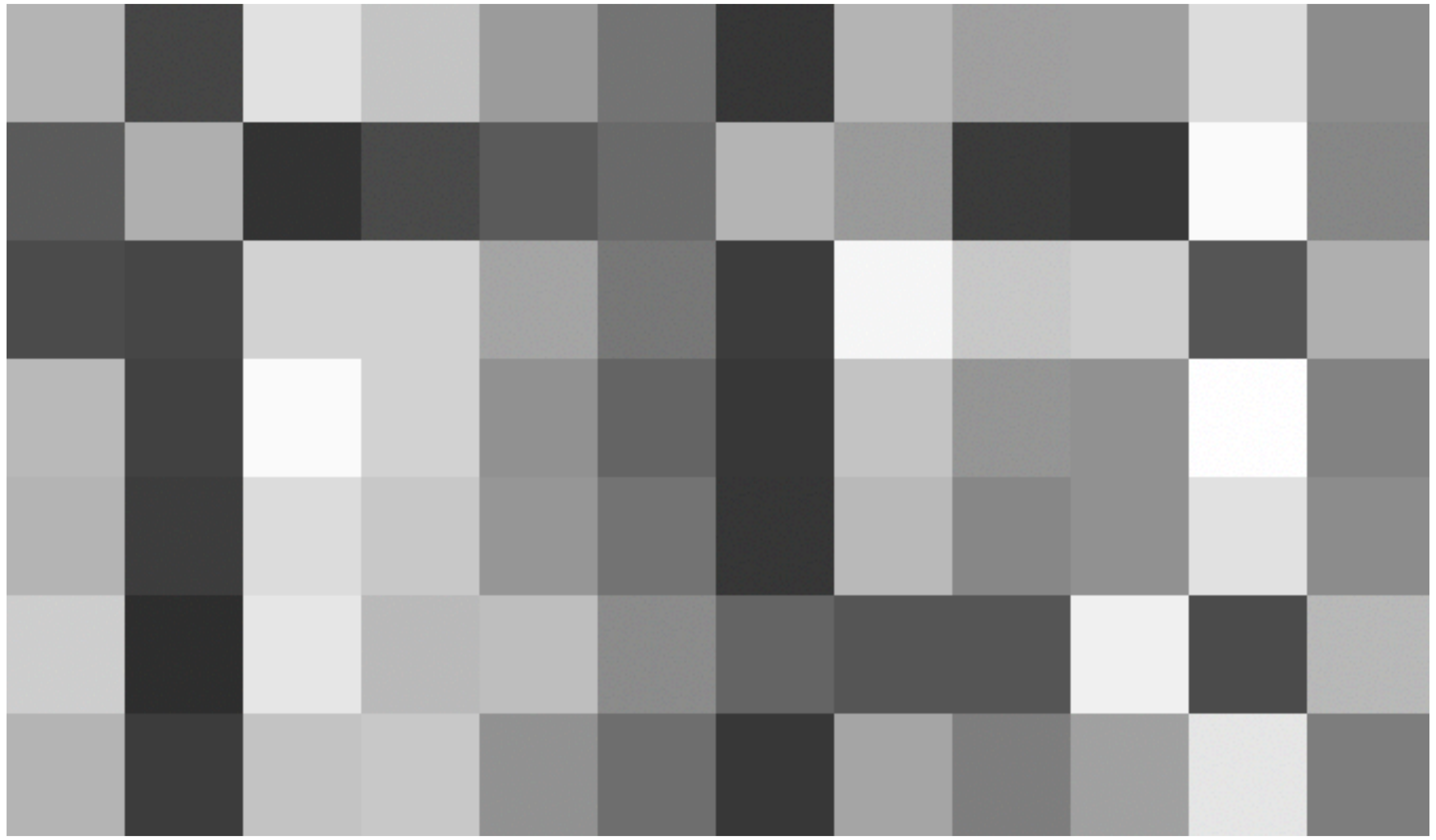

Structure like on the right unlikely in i.i.d. models. If practice looks like right-hand side, limits relevance of i.i.d. results.

Our Model: Smoothed Analysis

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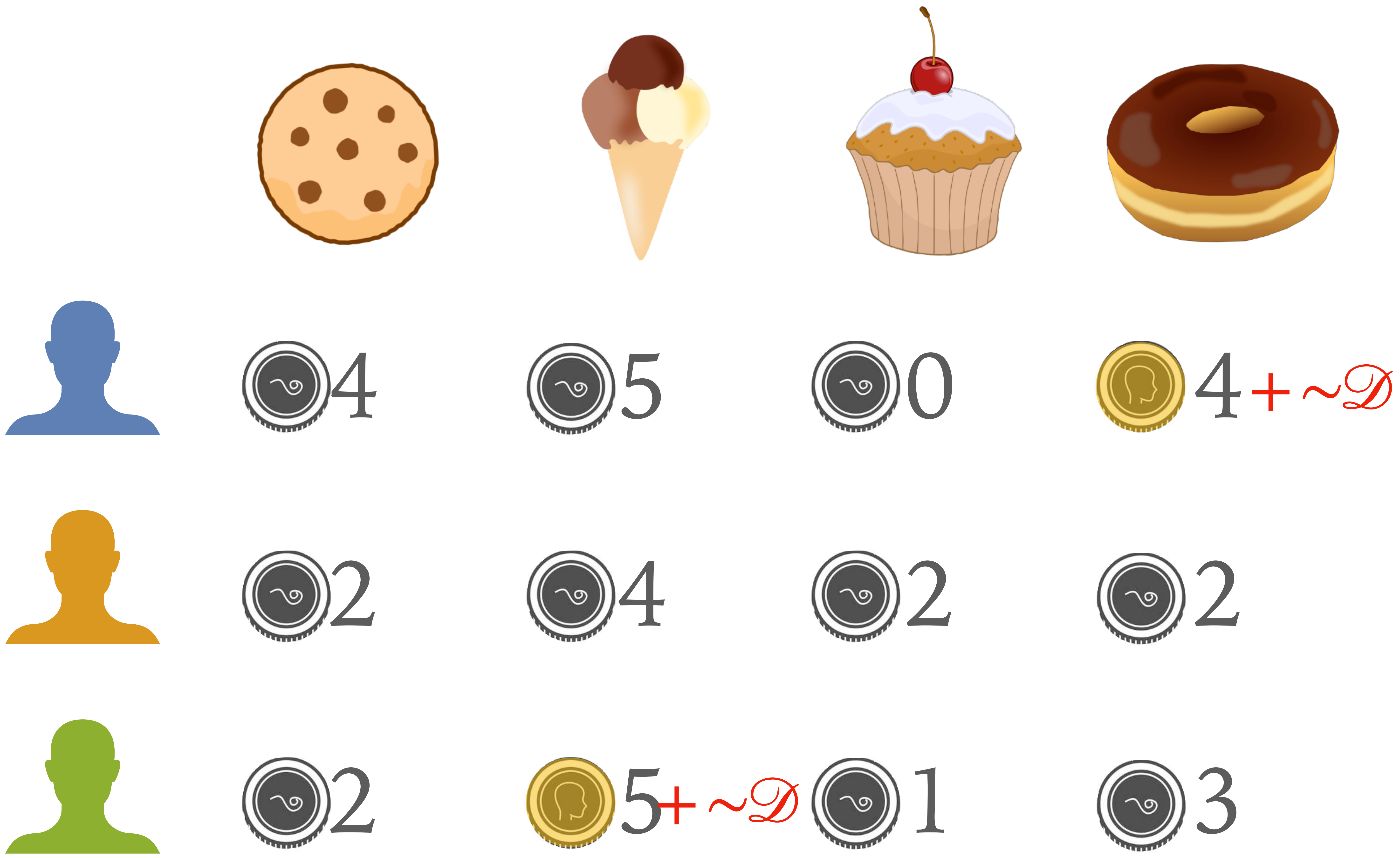
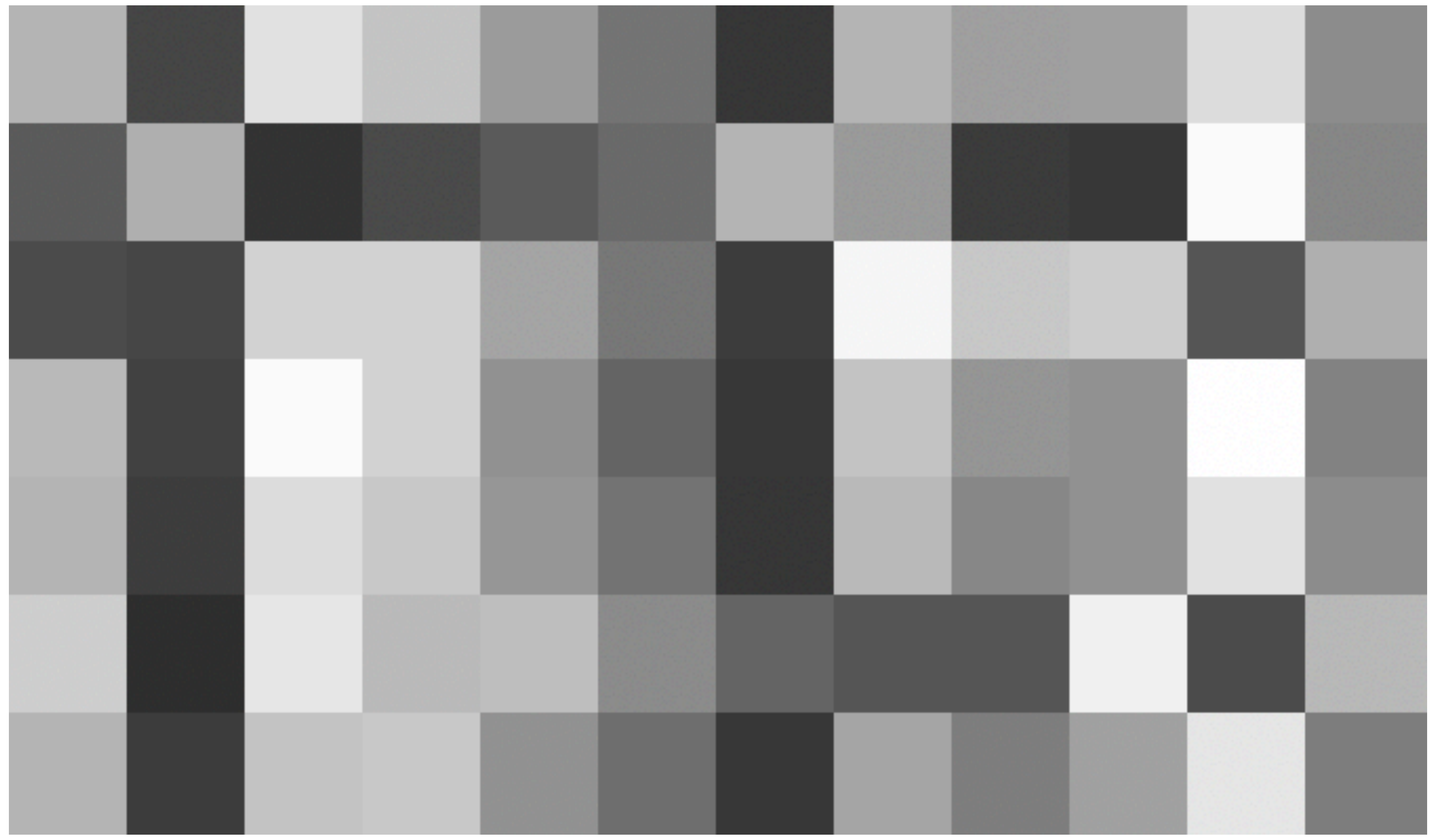


1. Start from worst-case *base values*.
2. For each entry, flip biased coin with probability $p \ll 1$.
3. Where coin comes up heads, “boost” value by amount $\sim \mathscr{D}$.

Our Model: Smoothed Analysis

base utilities



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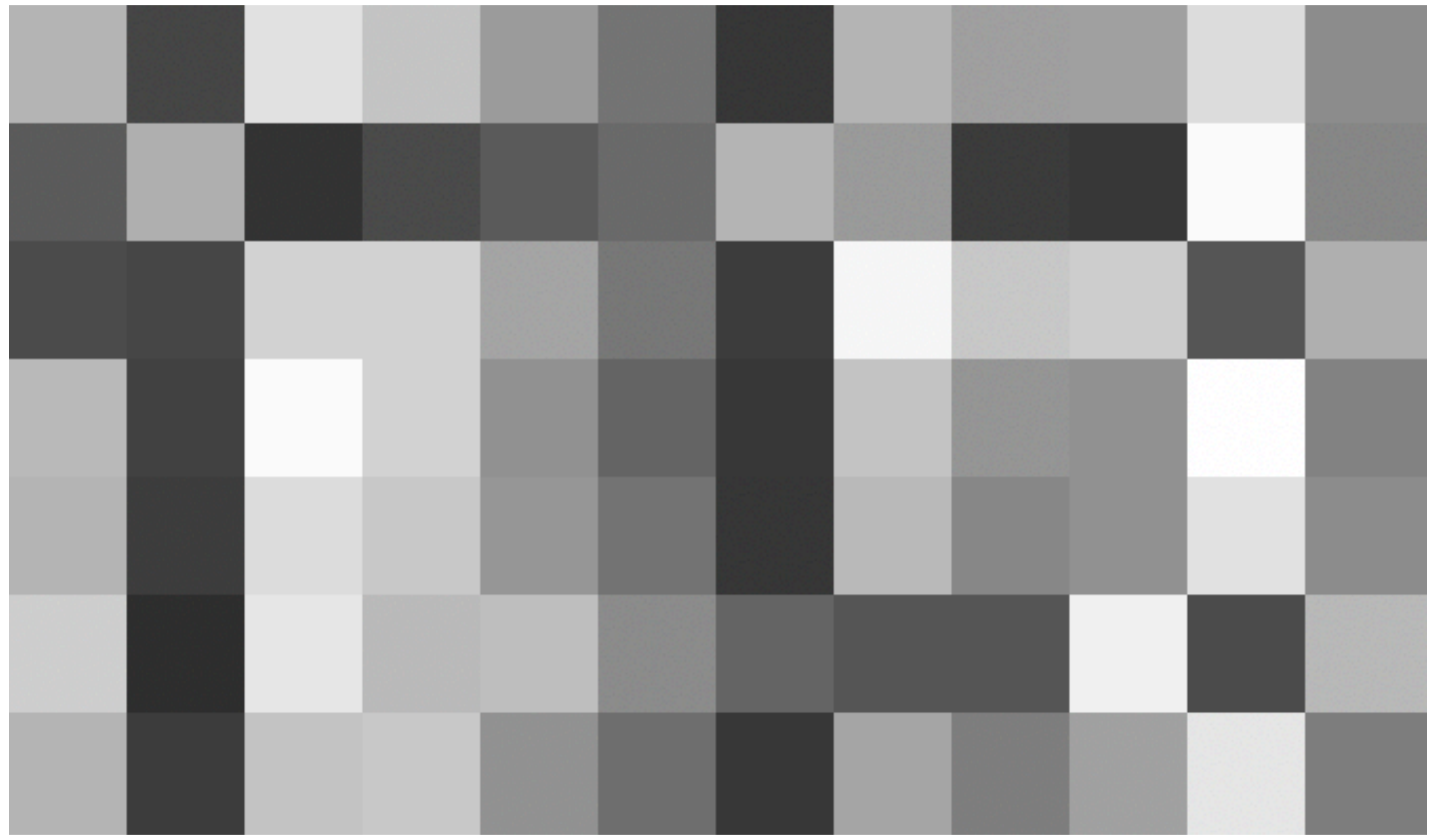

random
boosts

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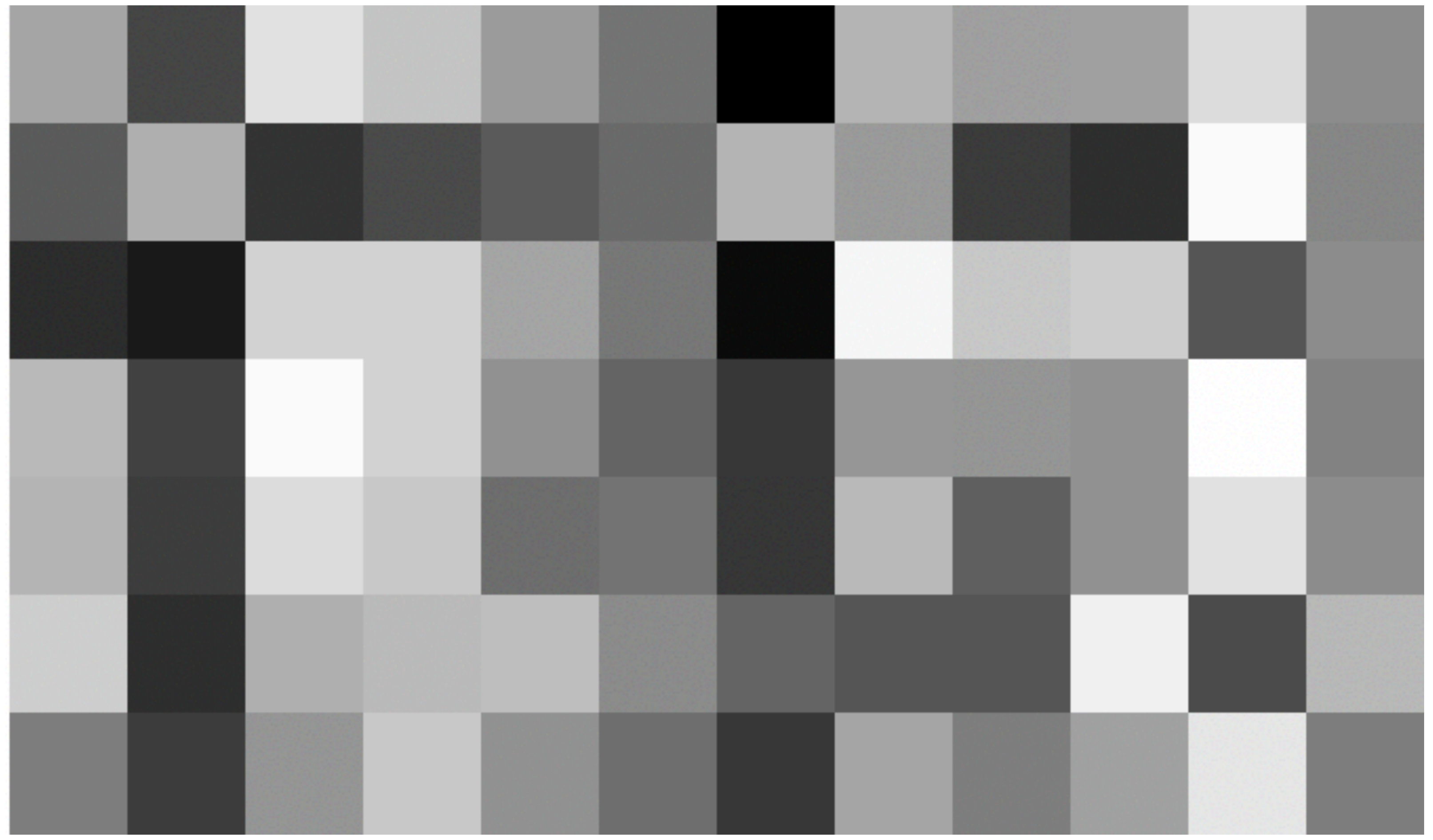

Theorem. If $m \gg n \log n$ and $p \gg \log n/m$, envy-free allocations exist in smoothed model with probability $\to 1$.

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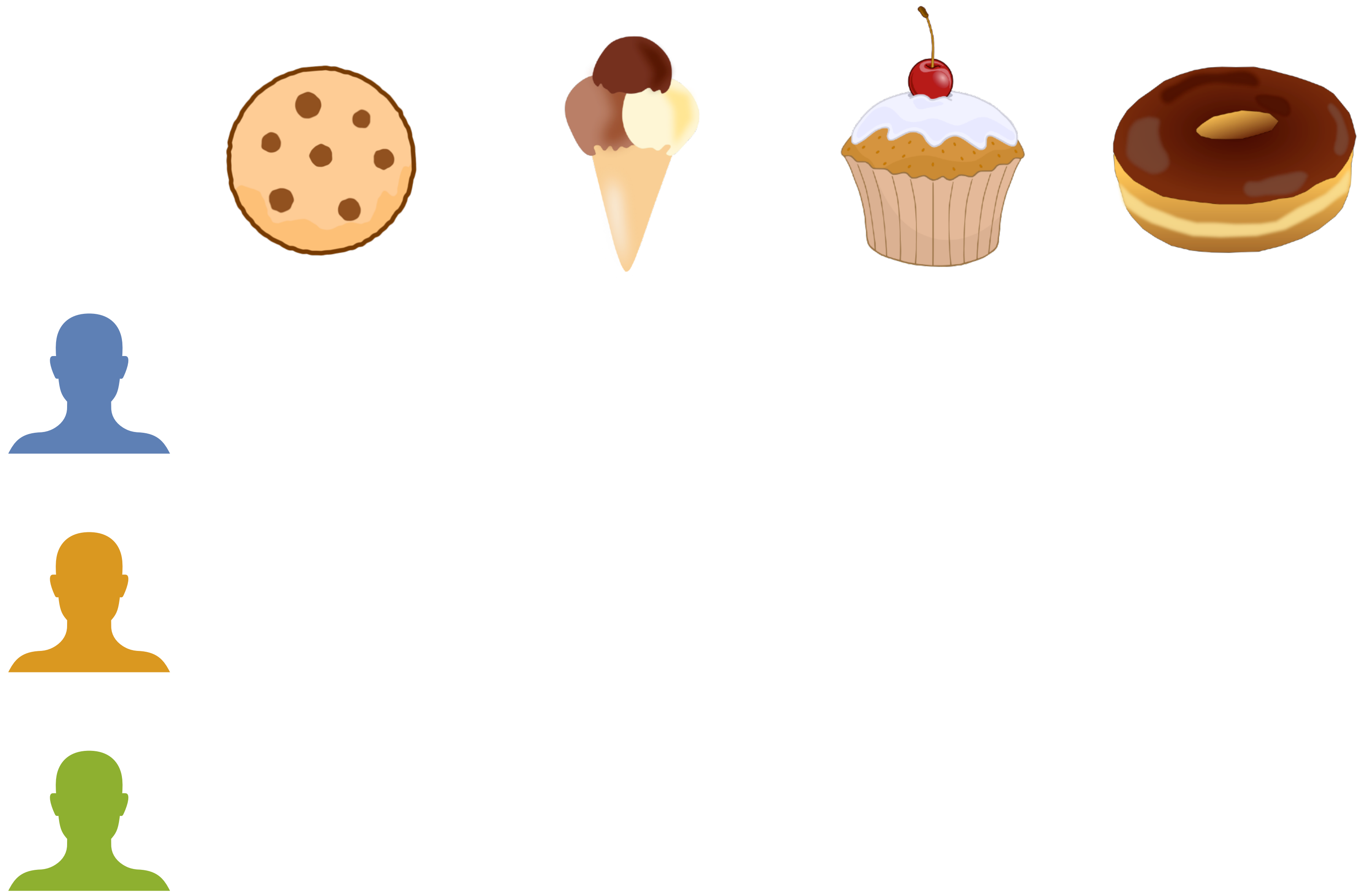
Algorithm for Finding EF Allocations w.h.p.



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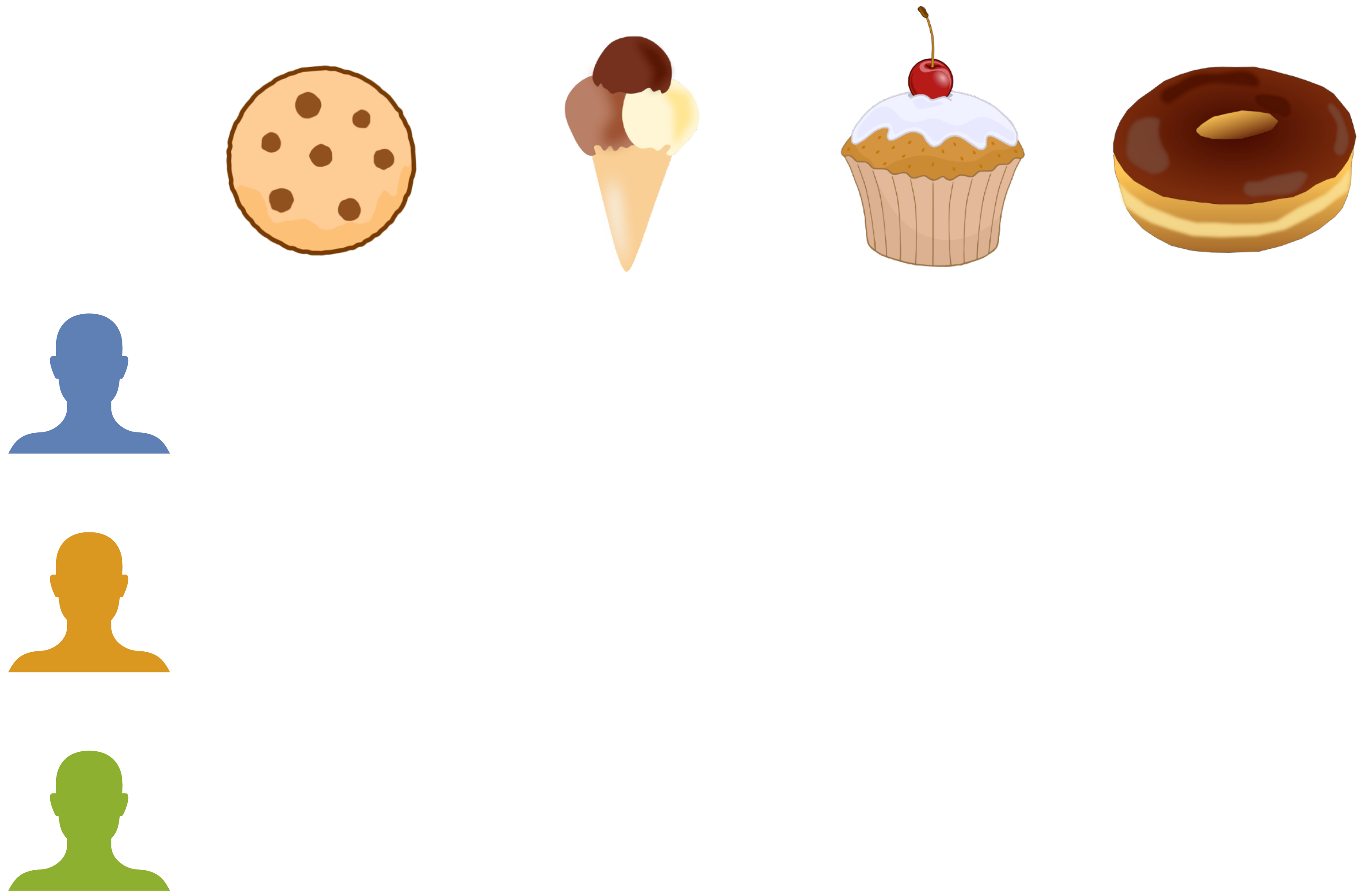

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- with LP, allocate ρ fractional items such that $t = \min_{\text{agents } i \neq j} u_i(A_i) - u_i(A_j)$ is maximized.

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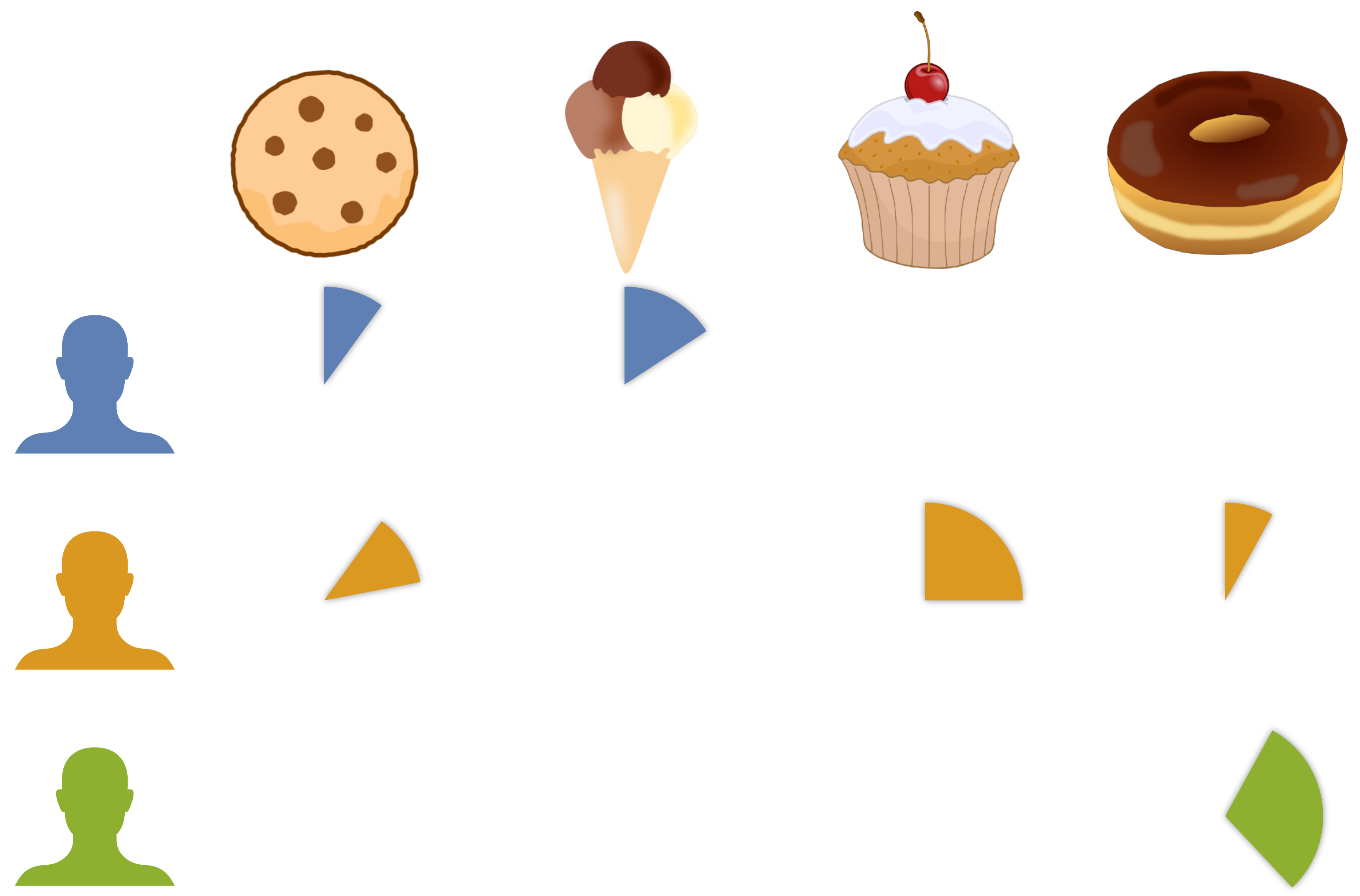

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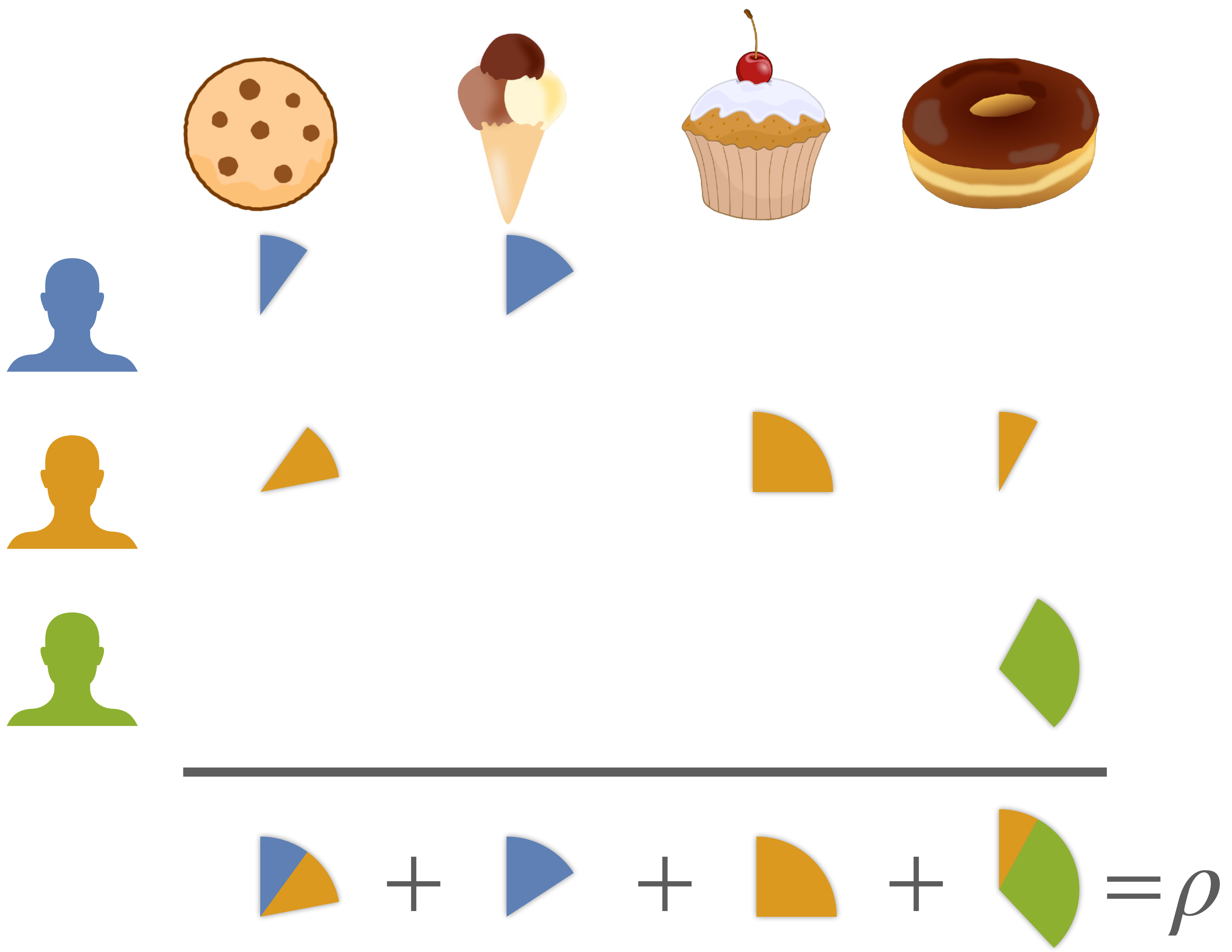

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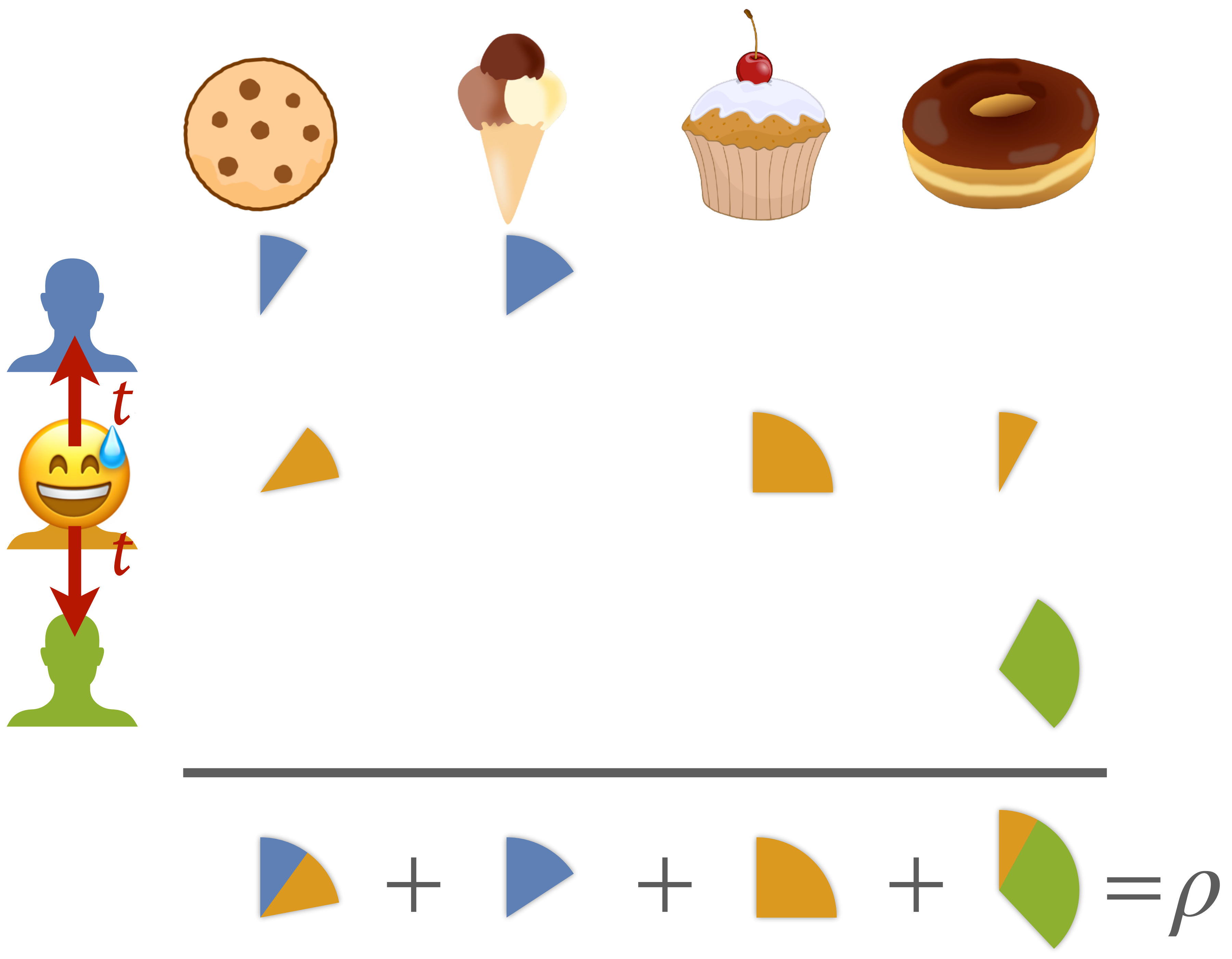

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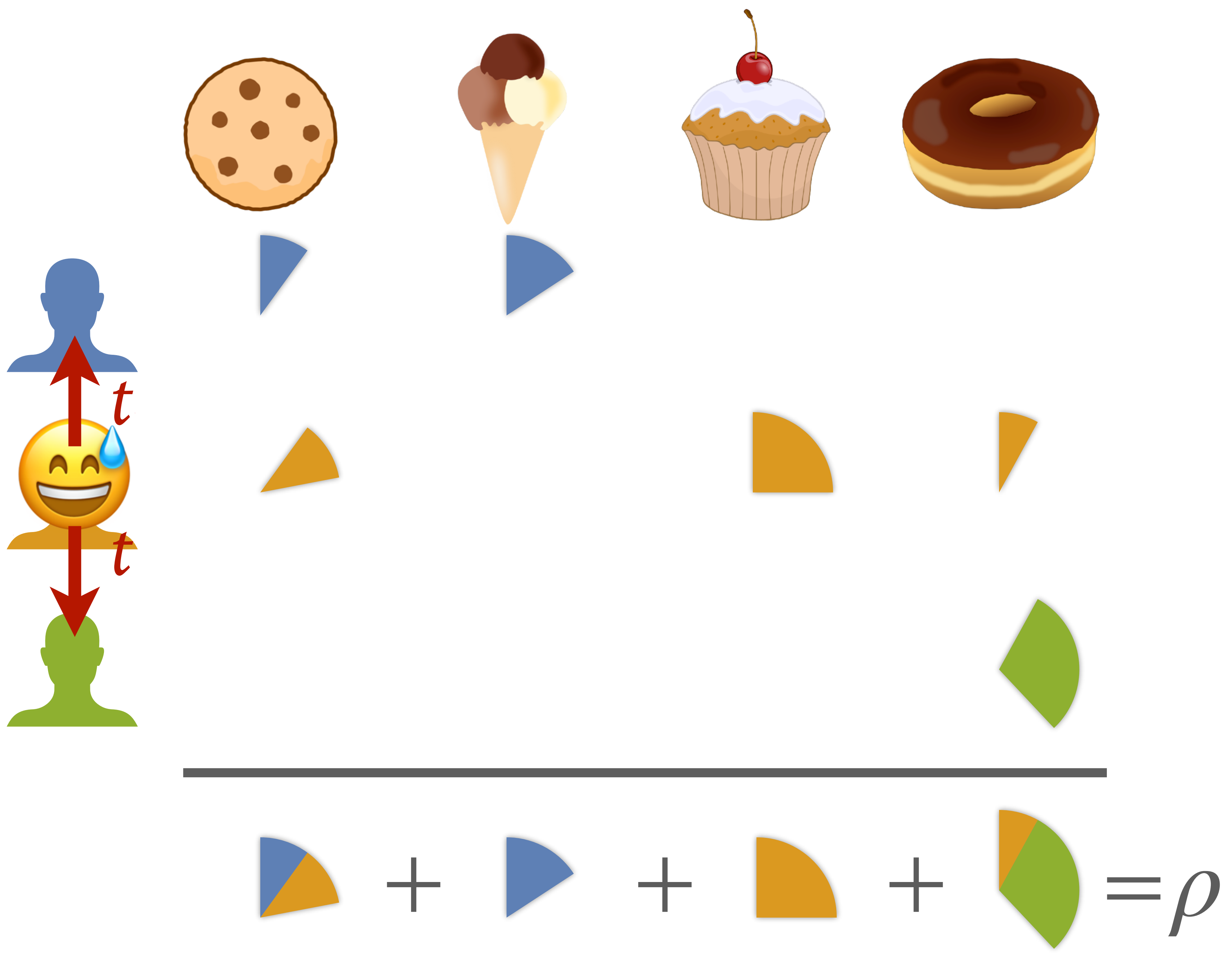

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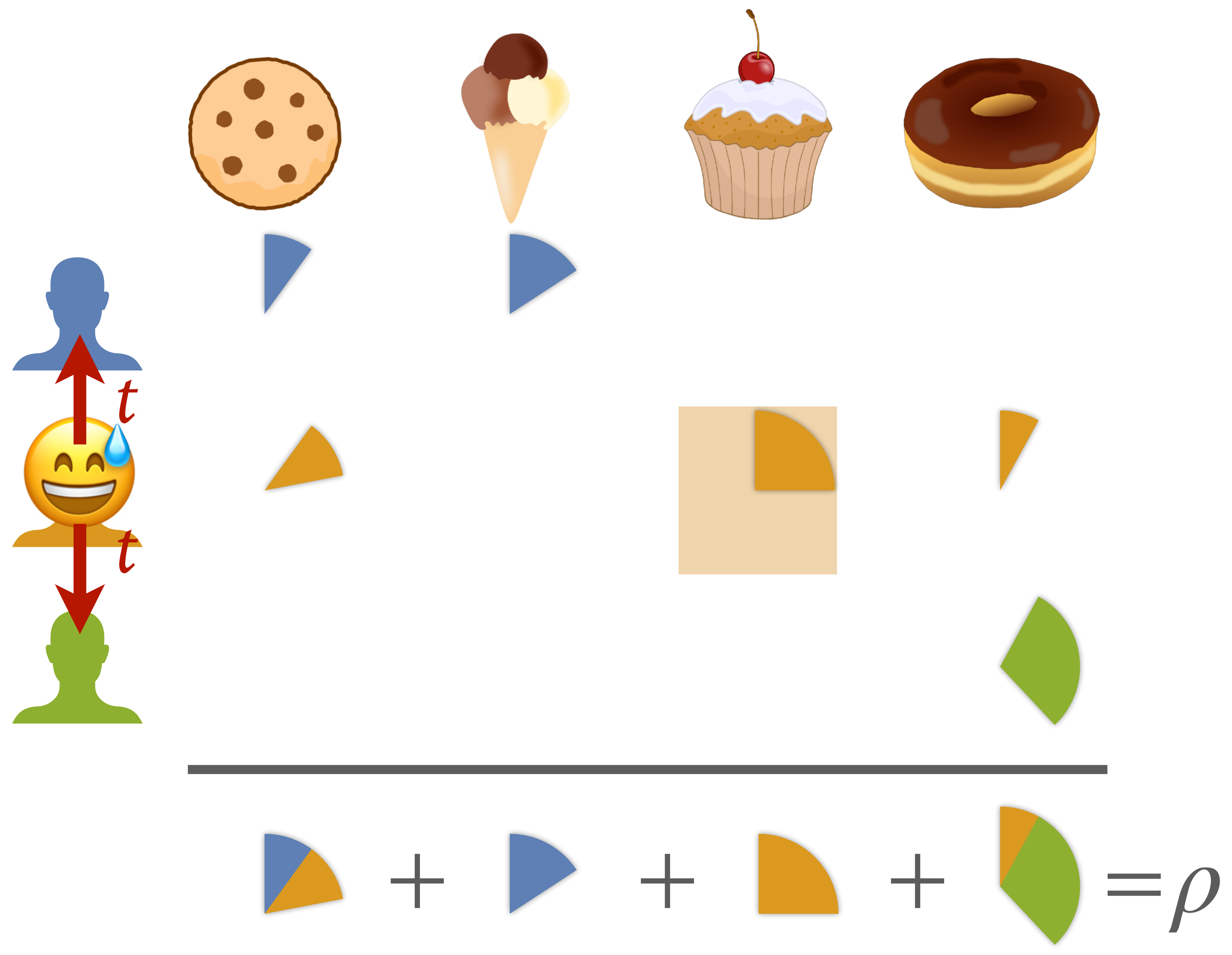

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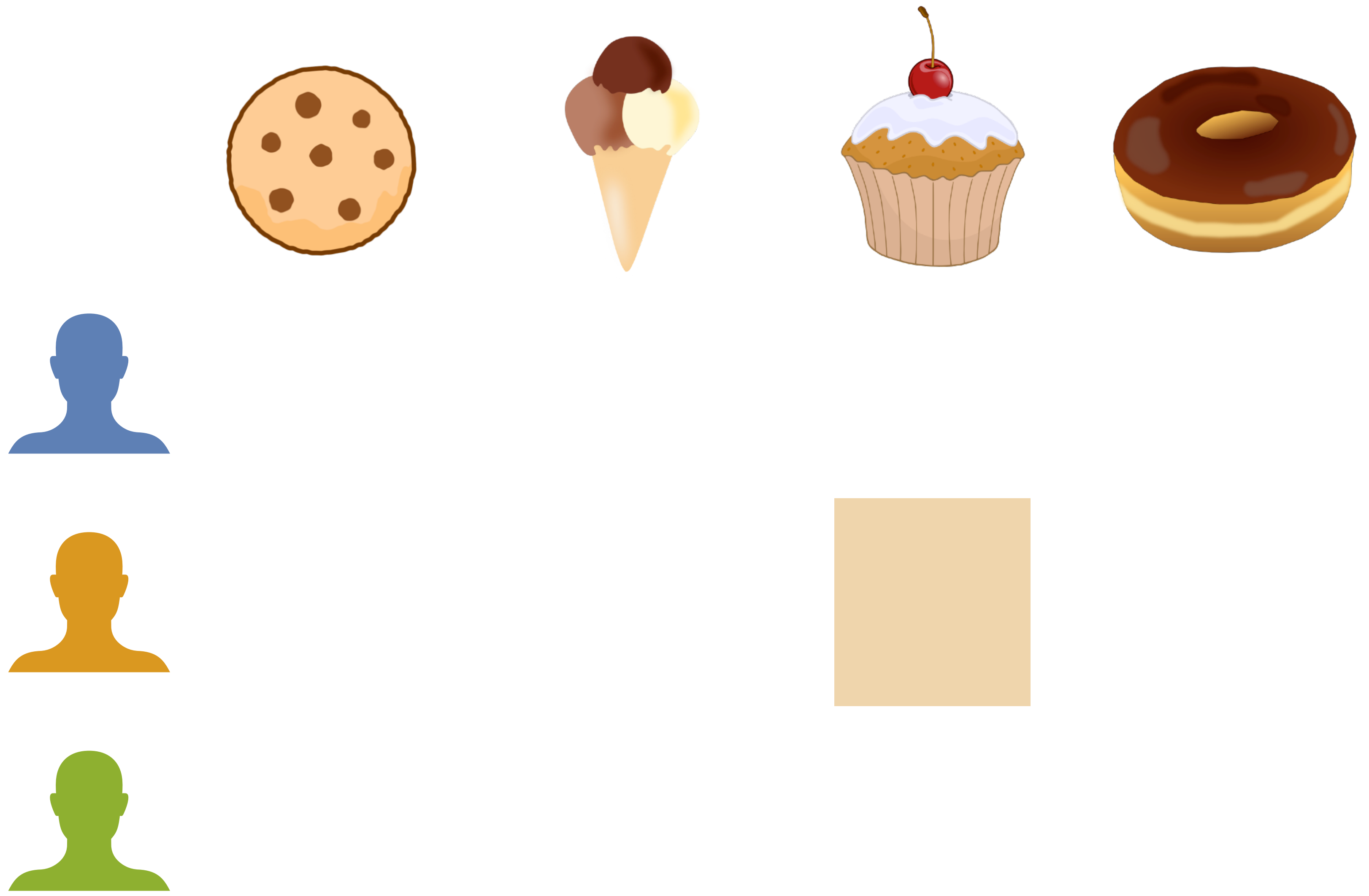

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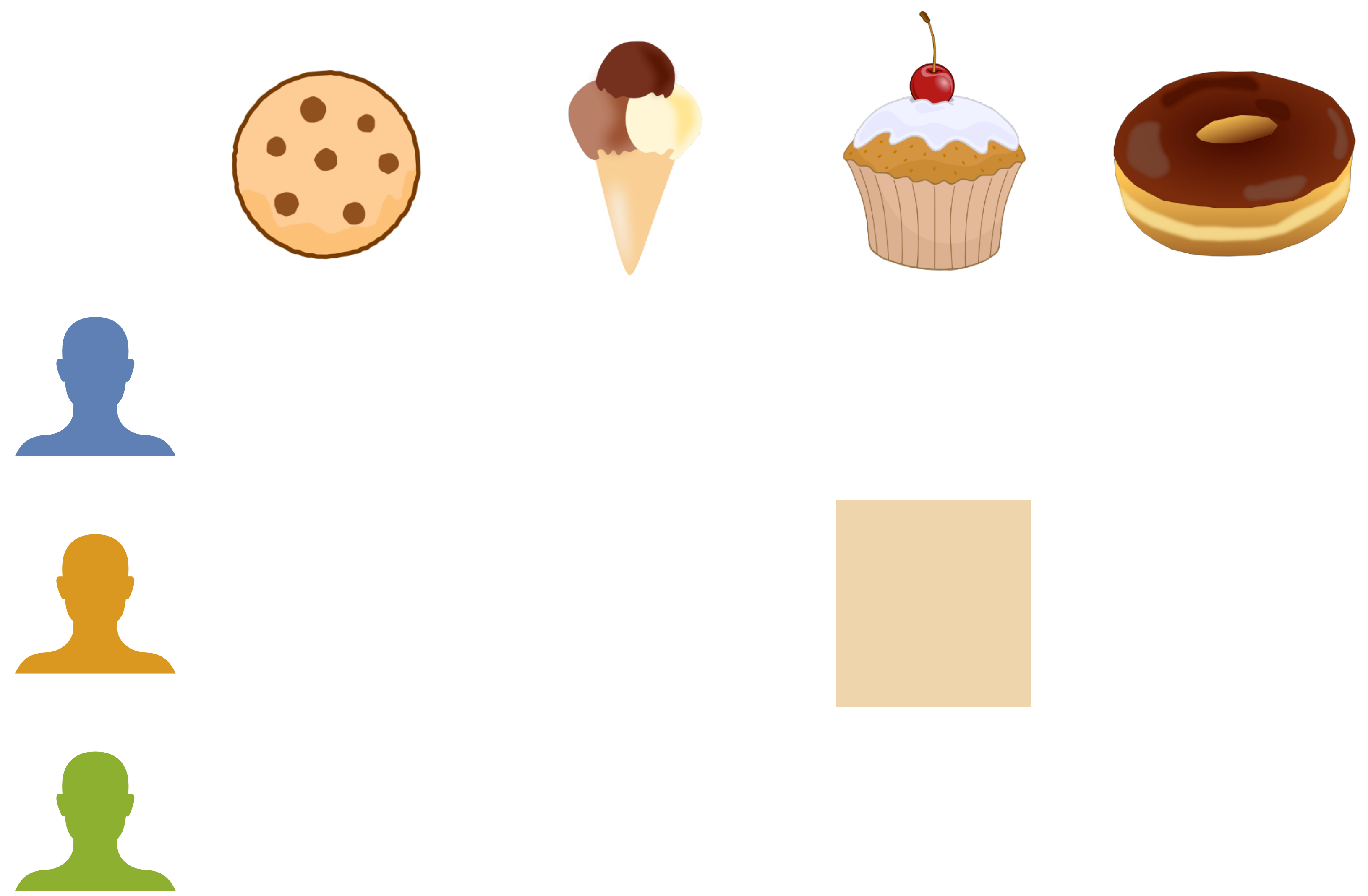

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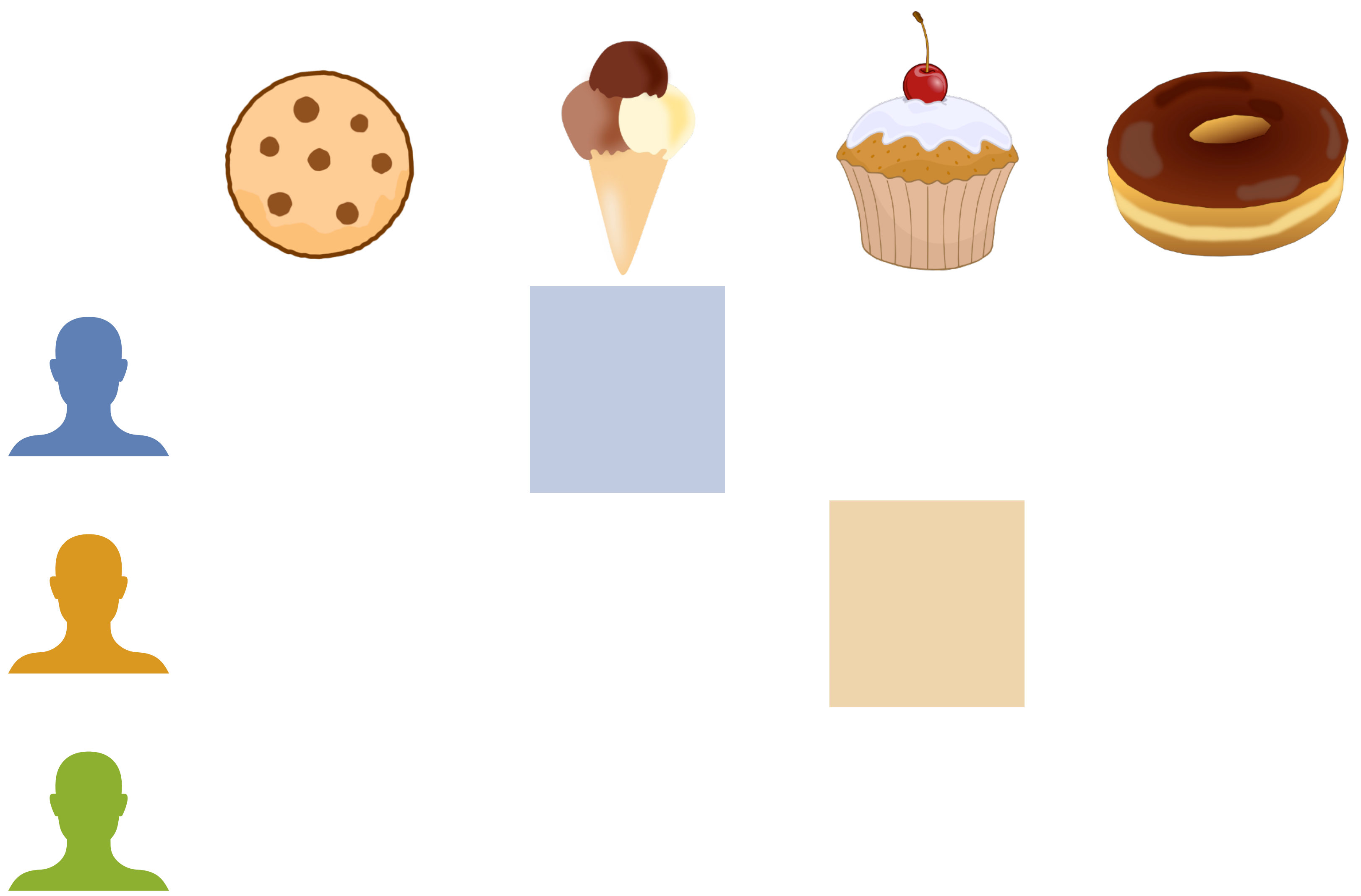

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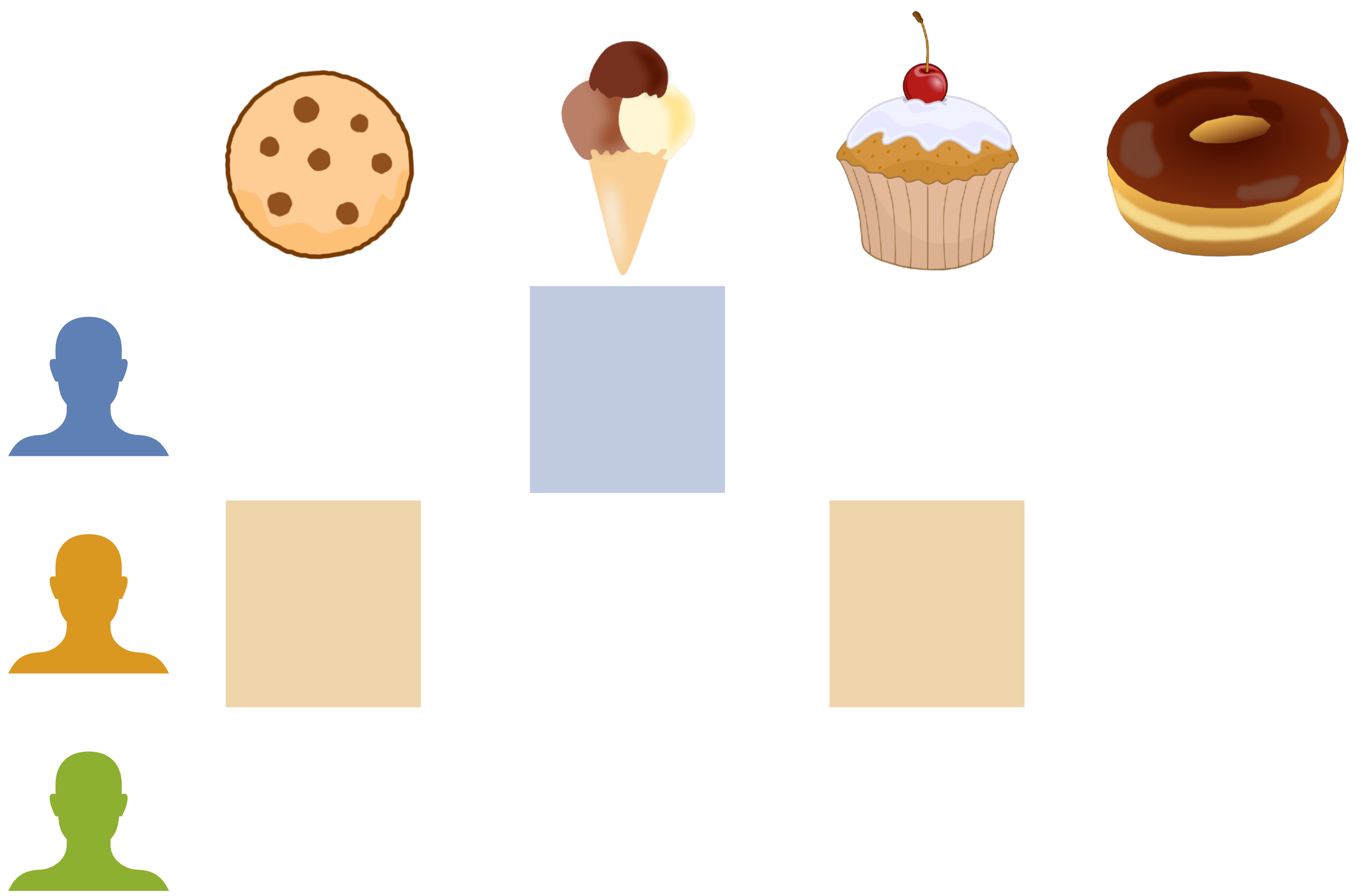

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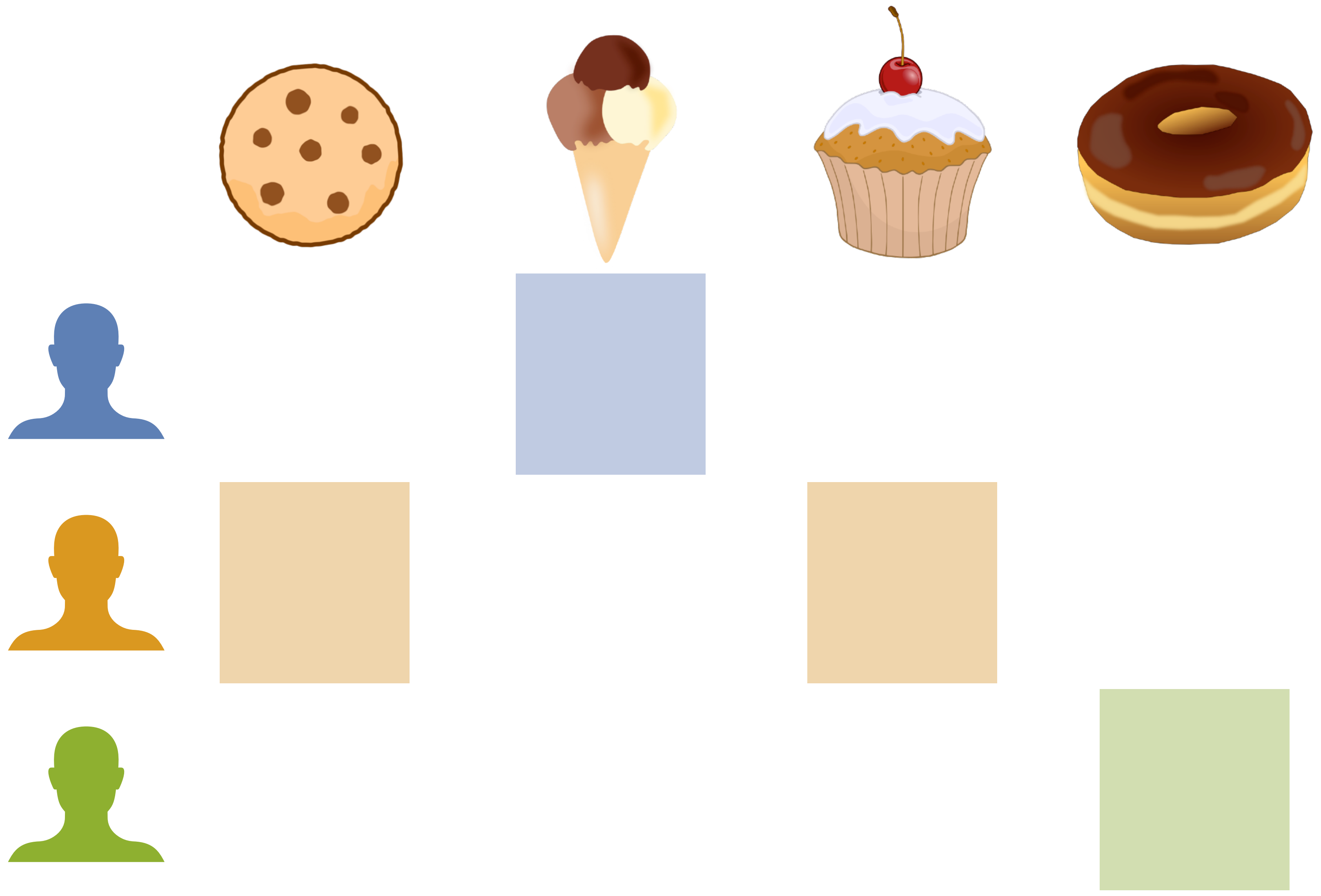

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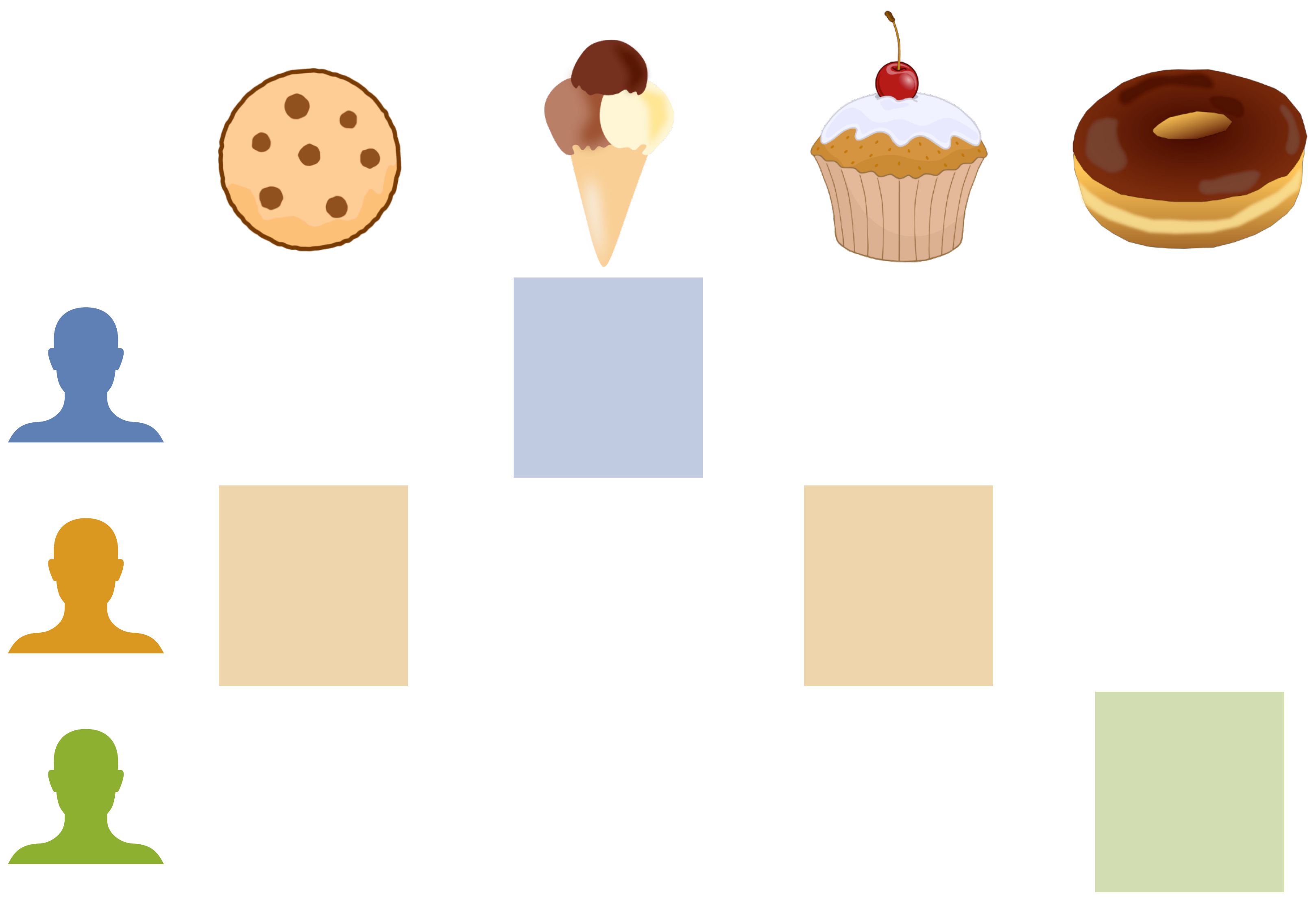

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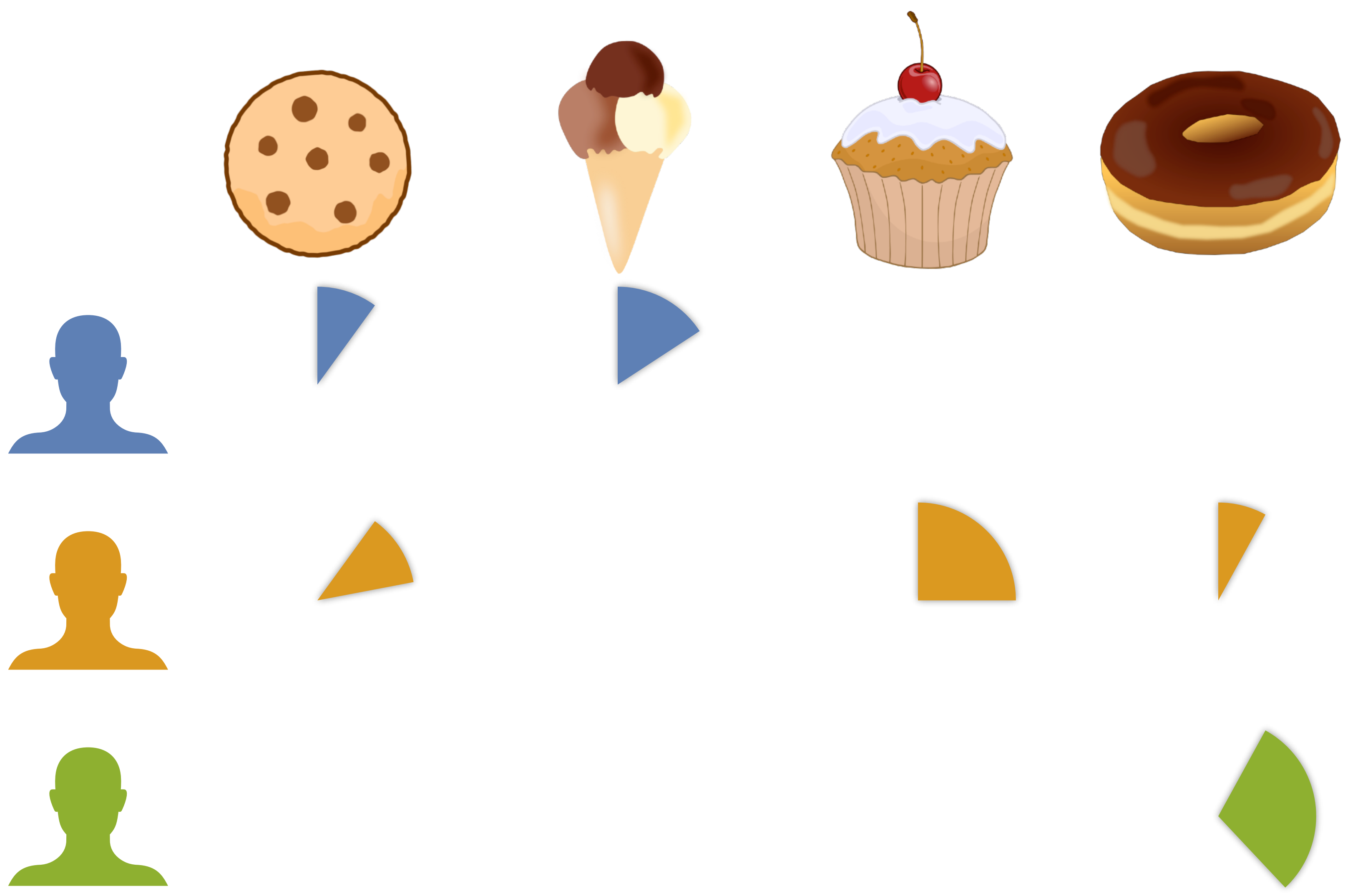

Overview of Results

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Check out our paper!
tinyurl.com/smoothEF