In This Apportionment Lottery, the House Always Wins

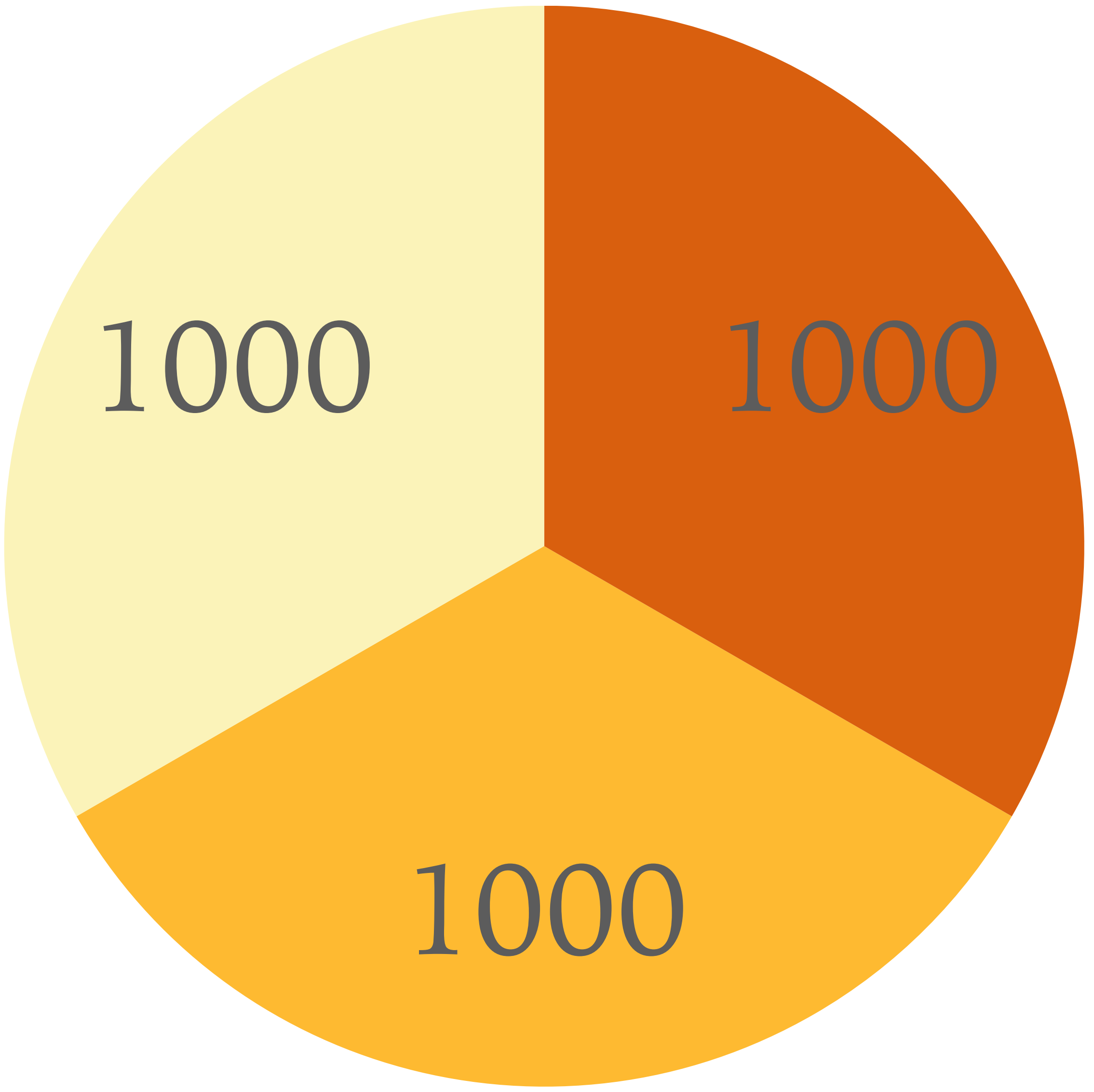
Paul Gölz, Dominik Peters & Ariel Procaccia

Deterministic Apportionment

legislature

Deterministic Apportionment

state populations

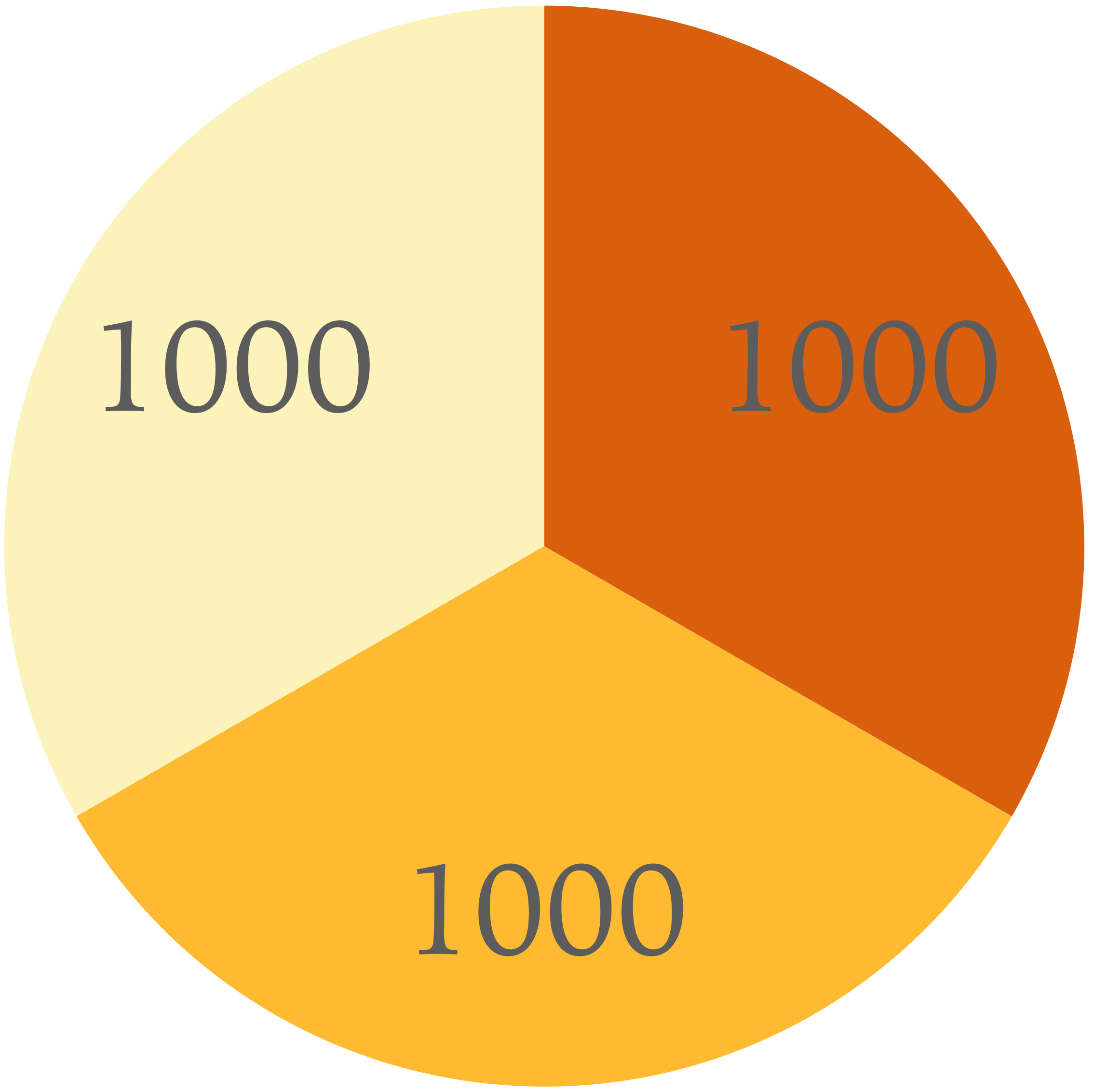


(1000, 1000, 1000)

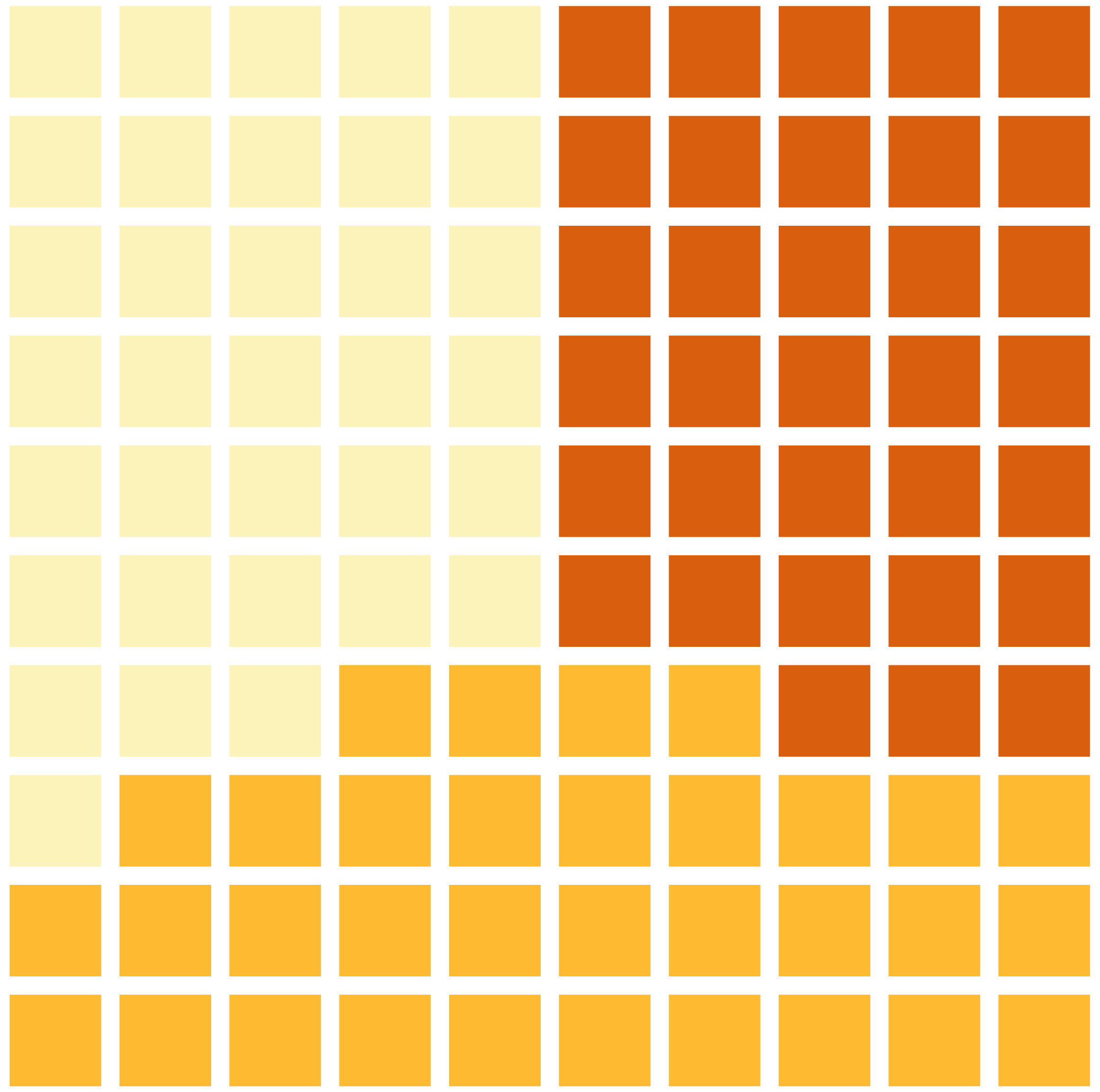
legislature

Deterministic Apportionment

state populations



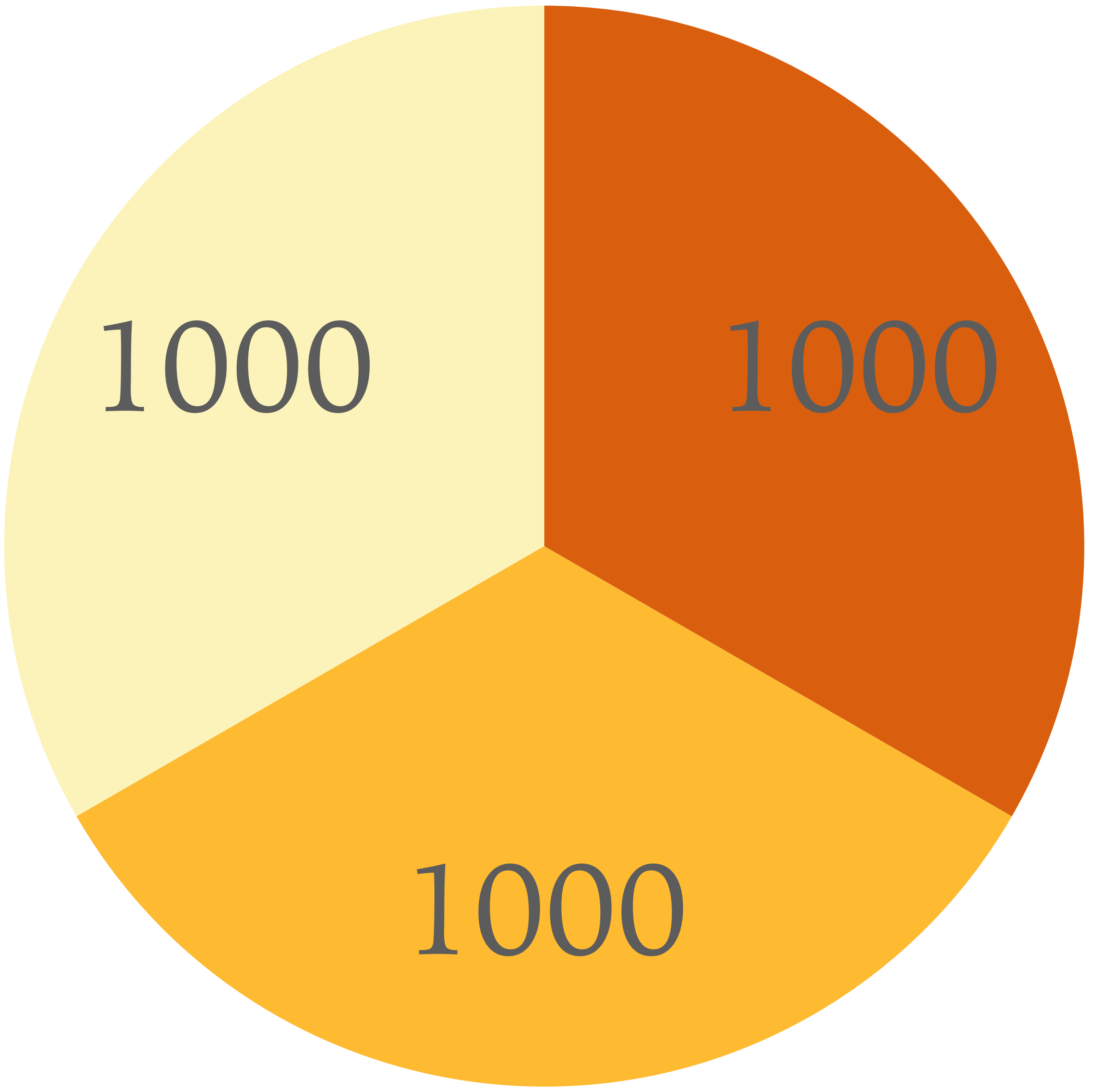
legislature



(1000, 1000, 1000)

Deterministic Apportionment

state populations



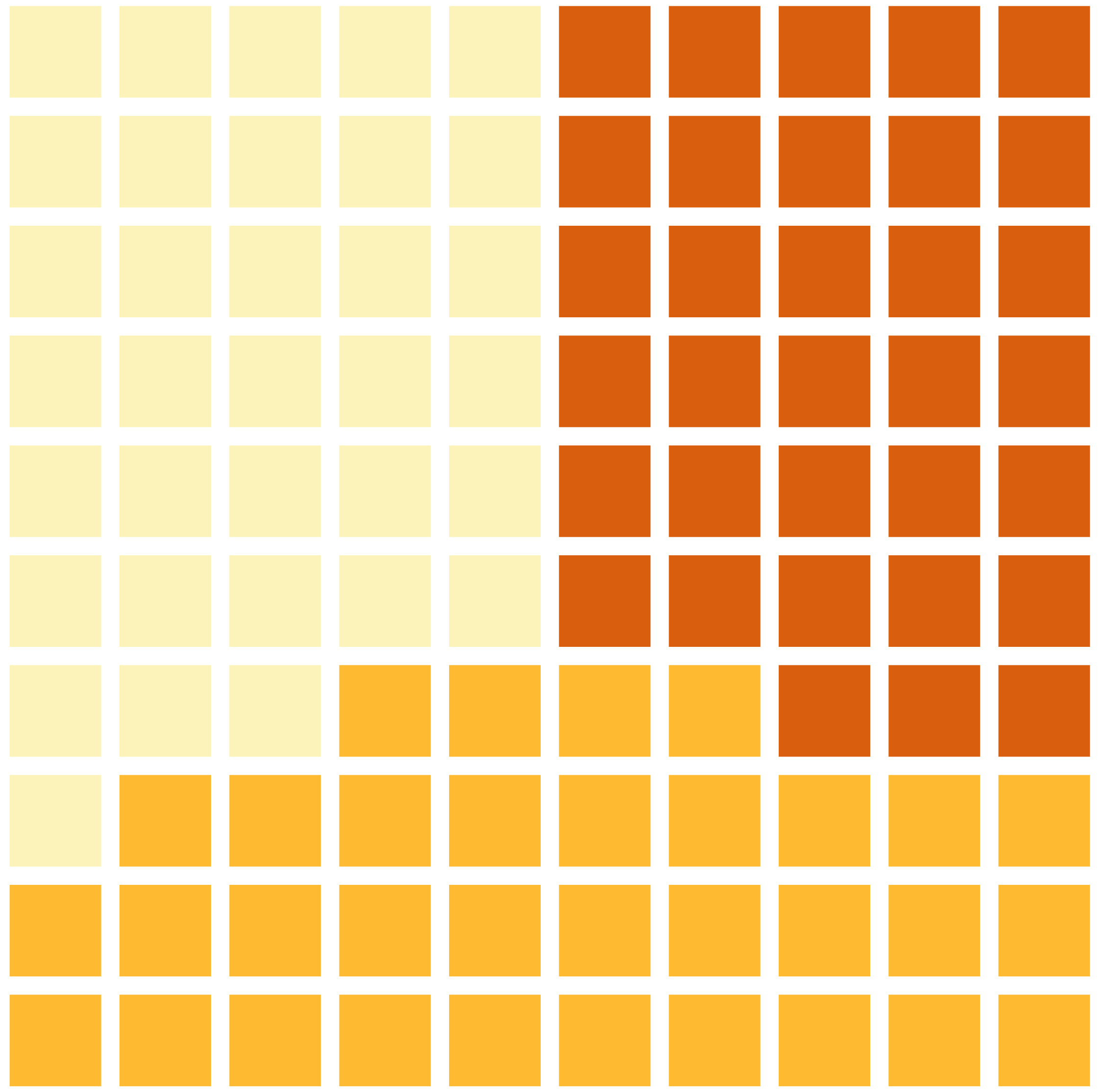
$(1000, 1000, 1000)$

$h = 100$

apportionment solution

$\longmapsto$

legislature



$(33, 33, 34)$

Axioms of Deterministic Apportionment

Quota: State's number of seats is floor or ceiling of its proportional share.

House monotonicity: If house size h increases, no state loses seats.

Population monotonicity.

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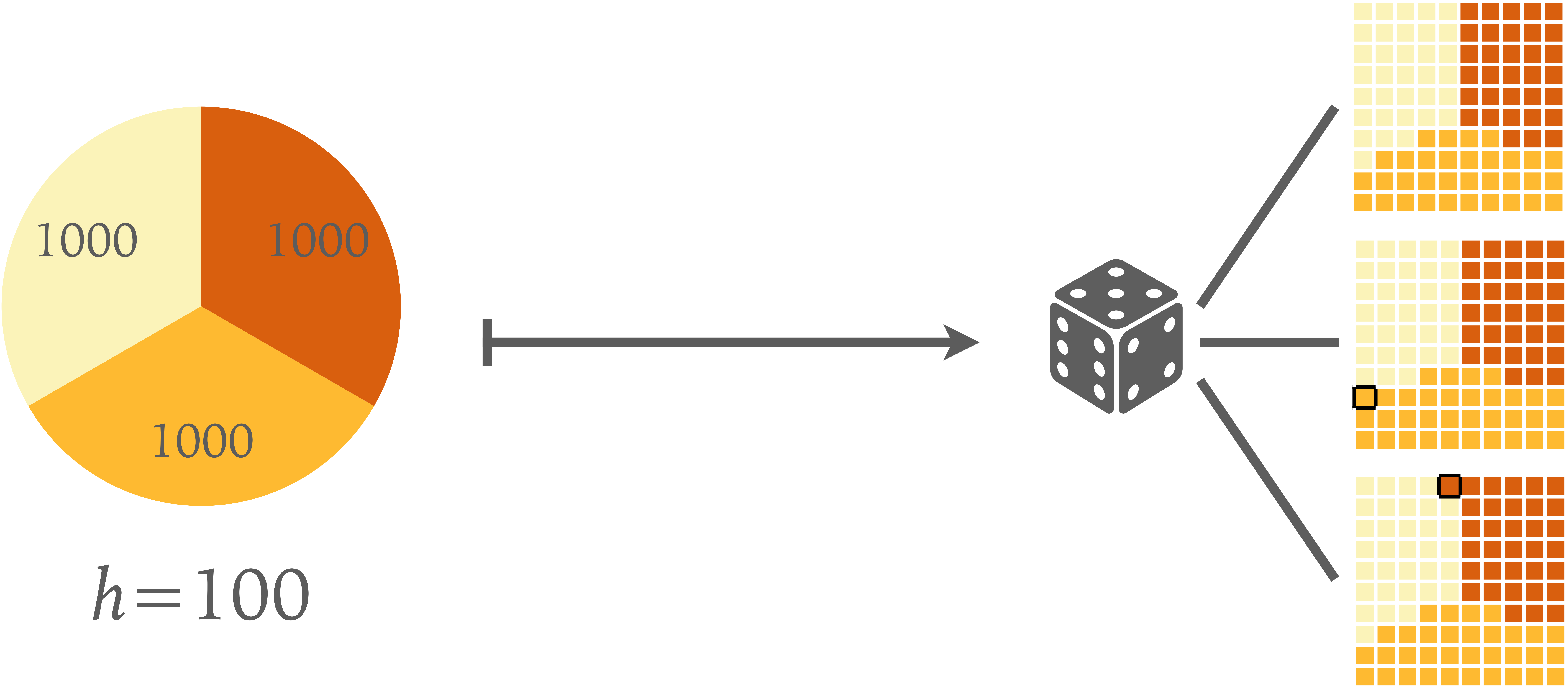
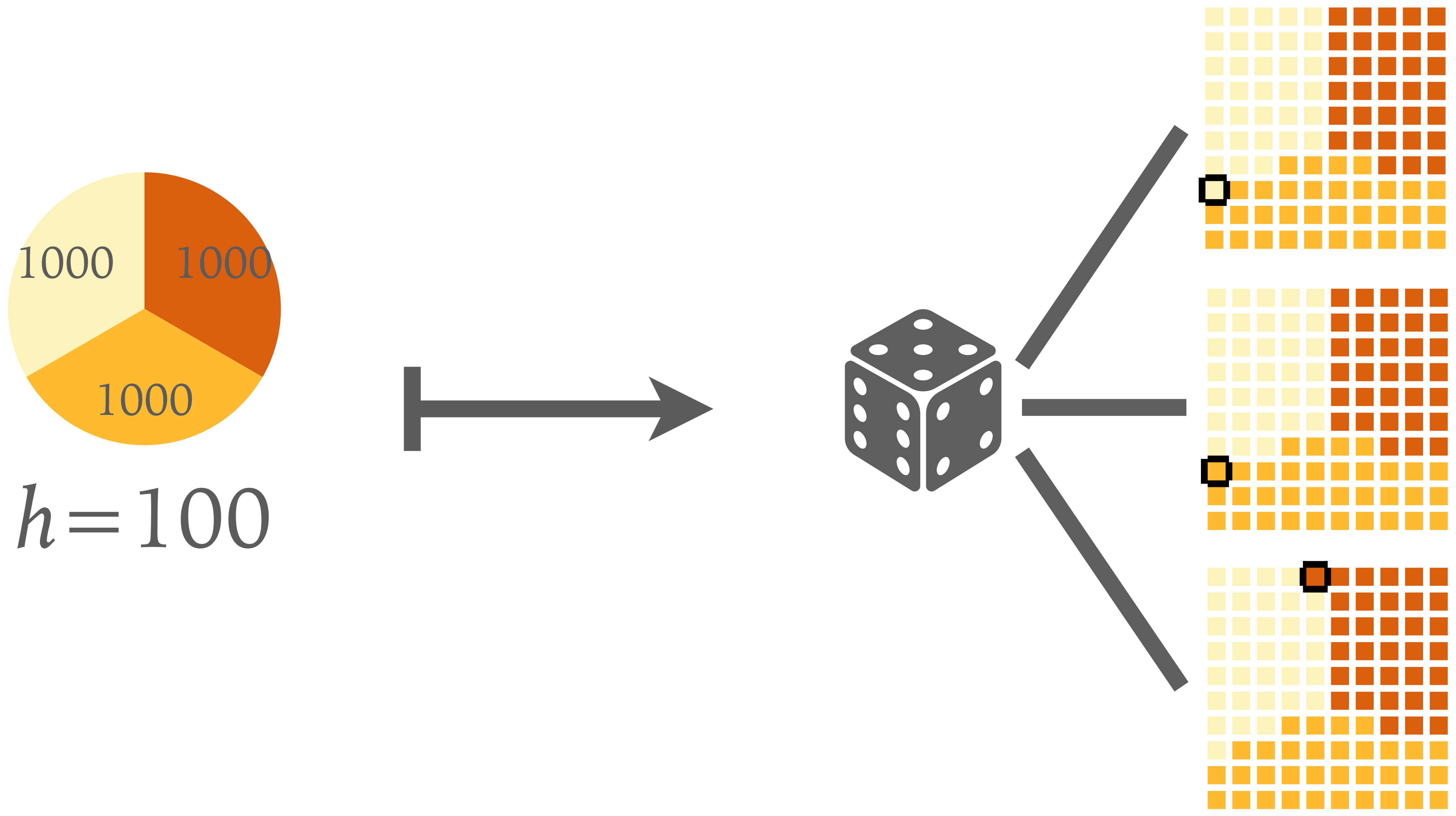
Elegant mathematics & political drama.

Randomized Apportionment [Grimmett04]



Grimmett, G. Stochastic Apportionment.
Am. Math. Mon. 111, 299–307 (2004).

Axioms Satisfied by Grimmett's Method



Ex-Ante Proportionality: State's expected number of seats equals its proportional share.

Quota (ex post).

Grimmett, G. Stochastic Apportionment. *Am. Math. Mon.* 111, 299–307 (2004).

Axioms Leave Too Many Candidates

Essentially, ex-ante proportionality & quota $\iff$ probability proportional to size sampling (πps):

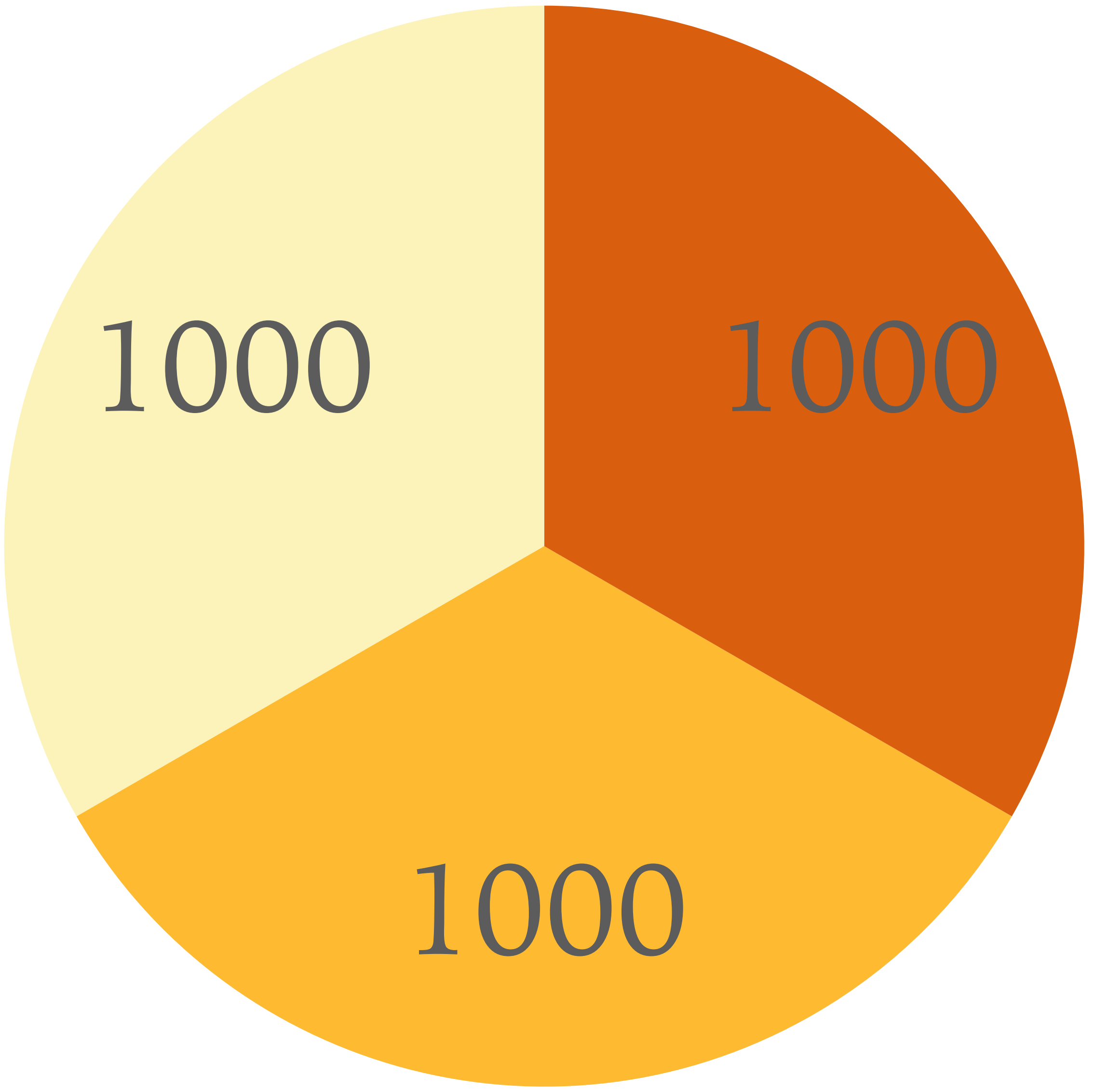
The following is a list of 50 *πpswor* procedures. A
each, and a basic reference is given. Descriptions

Procedure 1: Ordered Systematic Procedure; Madow (1949).
Procedure 2: Random Systematic Procedure; Goodman and Kish (1950).
Procedure 3: Grundy's Systematic Procedure; Grundy (1954).
Procedure 4: Yates-Grundy Draw-by-Draw Procedure; Yates and Grundy (1953).
Procedure 5: Yates-Grundy Rejective Procedure; Yates and Grundy (1953).
Procedure 6: Midzuno's Procedure; reported by Horvitz and Thompson (1952).
Procedure 7: Narain's Procedure; Narain (1951).
Procedure 8: Brewer's Procedure; Brewer (1963, 1975).
Procedure 9: Durbin's "Method I"; Durbin (1967).
Procedure 10: Durbin's "Grouped Method"; Durbin (1967).
Procedure 11: Rao-Sampford Procedure; Rao (1965), Sampford (1967).
Procedure 12: Durbin-Sampford Procedure; Sampford (1967).
Procedure 13: Fellegi's Procedure; Fellegi (1963).
Procedure 14: Carroll-Hartley Rejective Procedure; Carroll and Hartley (1964).
Procedure 15: Carroll-Hartley Draw-by-Draw Procedure; Carroll and Hartley (1964).
Procedure 16: Carroll-Hartley Whole Sample Procedure; Carroll and Hartley (1964).
Procedure 17: Durbin-Hanurav Procedure; Durbin (1953b); Hanurav (1966, 1967).
Procedure 18: Hanurav's Scheme B-A'; Hanurav (1967).
Procedure 19: Hanurav-Vijayan Procedure; Hanurav (1967); Vijayan (1968).
Procedure 20: Raj's Variance Minimization Procedure; Raj (1956b).
Procedure 21: Hanurav's Simple Functional Procedure; Hanurav (1962a).
Procedure 22: Hanurav's Modified Functional Procedure; Hanurav (1962a).
Procedure 23: Hanurav's Double Functional Procedure; Hanurav (1962a).

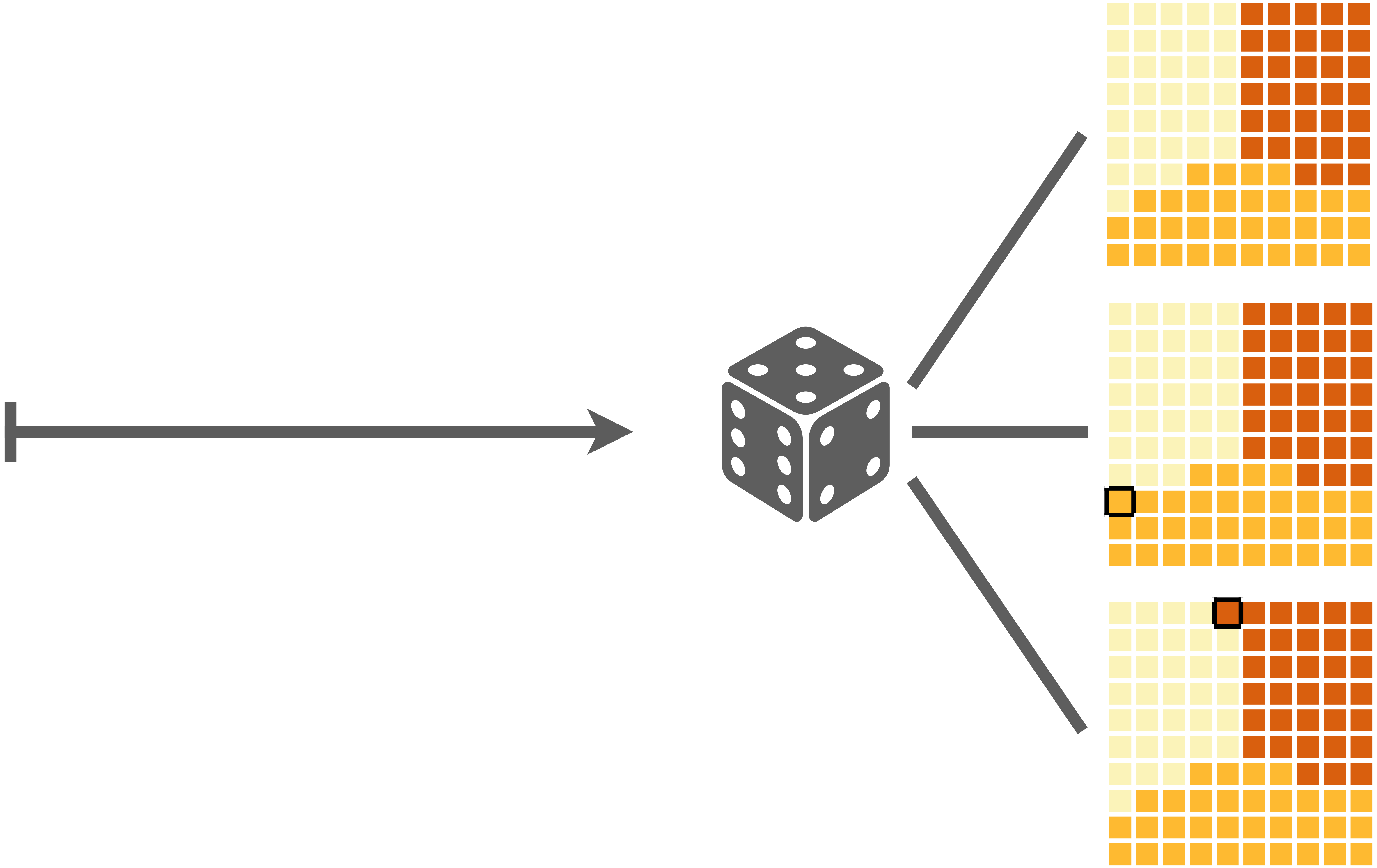
Procedure 24: Hanurav's Sequential Procedure; Hanurav (1962a).
Procedure 25: Rao-Hartley-Cochran Procedure; Rao, Hartley and Cochran (1962).
Procedure 26: Stevens' Procedure; Stevens (1958).
Procedure 27: Poisson Sampling; Hajek (1964b).
Procedure 28: Hajek's "Method I"; Hajek (1964b).
Procedure 29: Hajek's "Method II"; Hajek (1964b).
Procedure 30: Hajek's "Method III"; Hajek (1964b).
Procedure 31: Hajek's "Method IV"; Hajek (1964b).
Procedure 32: Deming's Systematic Procedure; Deming (1960).
Procedure 33: Variance Estimator Optimization Procedure; Brewer and Hanif (1969a).
Procedure 34: Jessen's "Method 1"; Jessen (1969).
Procedure 35: Jessen's "Method 2"; Jessen (1969).
Procedure 36: Jessen's "Method 3"; Jessen (1969).
Procedure 37: Jessen's "Method 4"; Jessen (1969).
Procedure 38: Modified Poisson Sampling; Ogus and Clark (1971).
Procedure 39: Collocated Sampling; Brewer, Early and Hanif (1980).
Procedure 40: Das-Mohanty Procedure; Das and Mohanty (1973).
Procedure 41: Mukhopadhyay's Procedure; Mukhopadhyay (1972).
Procedure 42: Sinha's Extension Procedure; Sinha (1973).
Procedure 43: Sinha's Reduction Procedure; Sinha (1973).
Procedure 44: Chaudhuri's Procedure; Chaudhuri (1976).
Procedure 45: Lahiri's Procedure; Lahiri (1951).
Procedure 46: Ikeda-Midzuno Procedure; Midzuno (1952).
Procedure 47: Fuller's "Scheme B"; Fuller (1971).
Procedure 48: Singh's Procedure; Singh (1978).
Procedure 49: Choudhry's Procedure; Choudhry (1979).
Procedure 50: Chromy's Procedure; Chromy (1979).

Brewer, K. R. W. & Hanif, M. Sampling with Unequal Probabilities. (Springer, 1983).

Grimmett's Definition of a Method



$h = 100$



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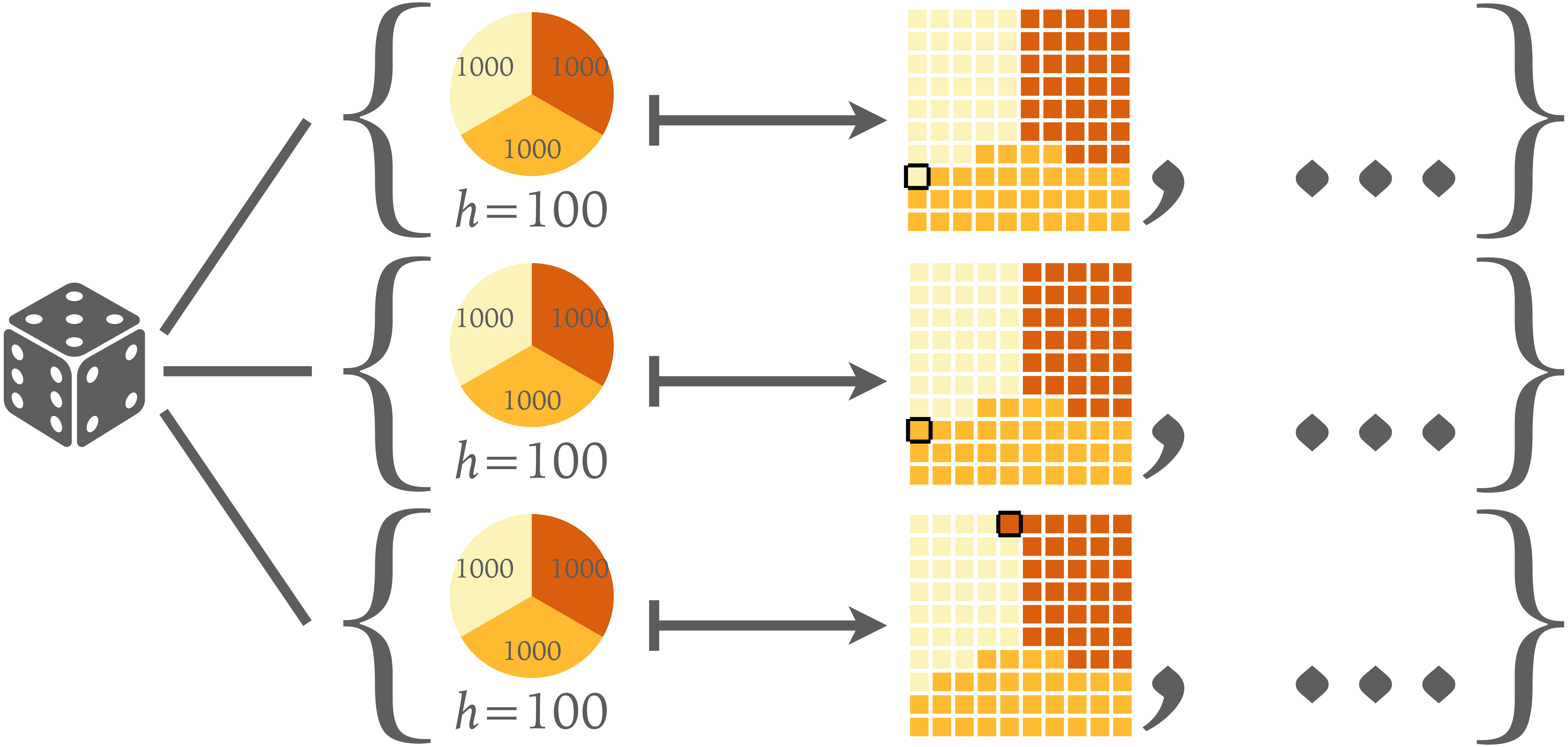
Our Definition of a Method



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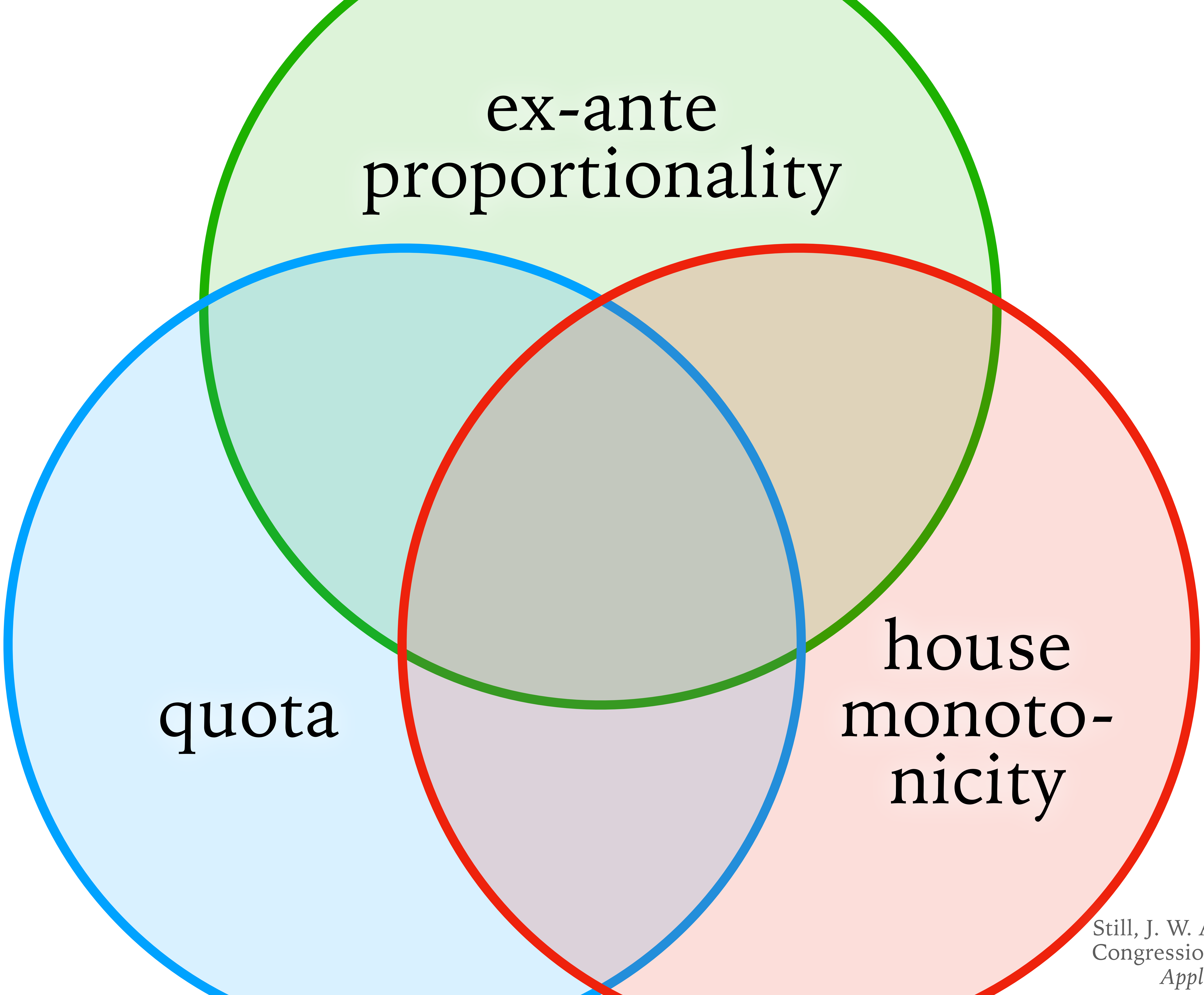
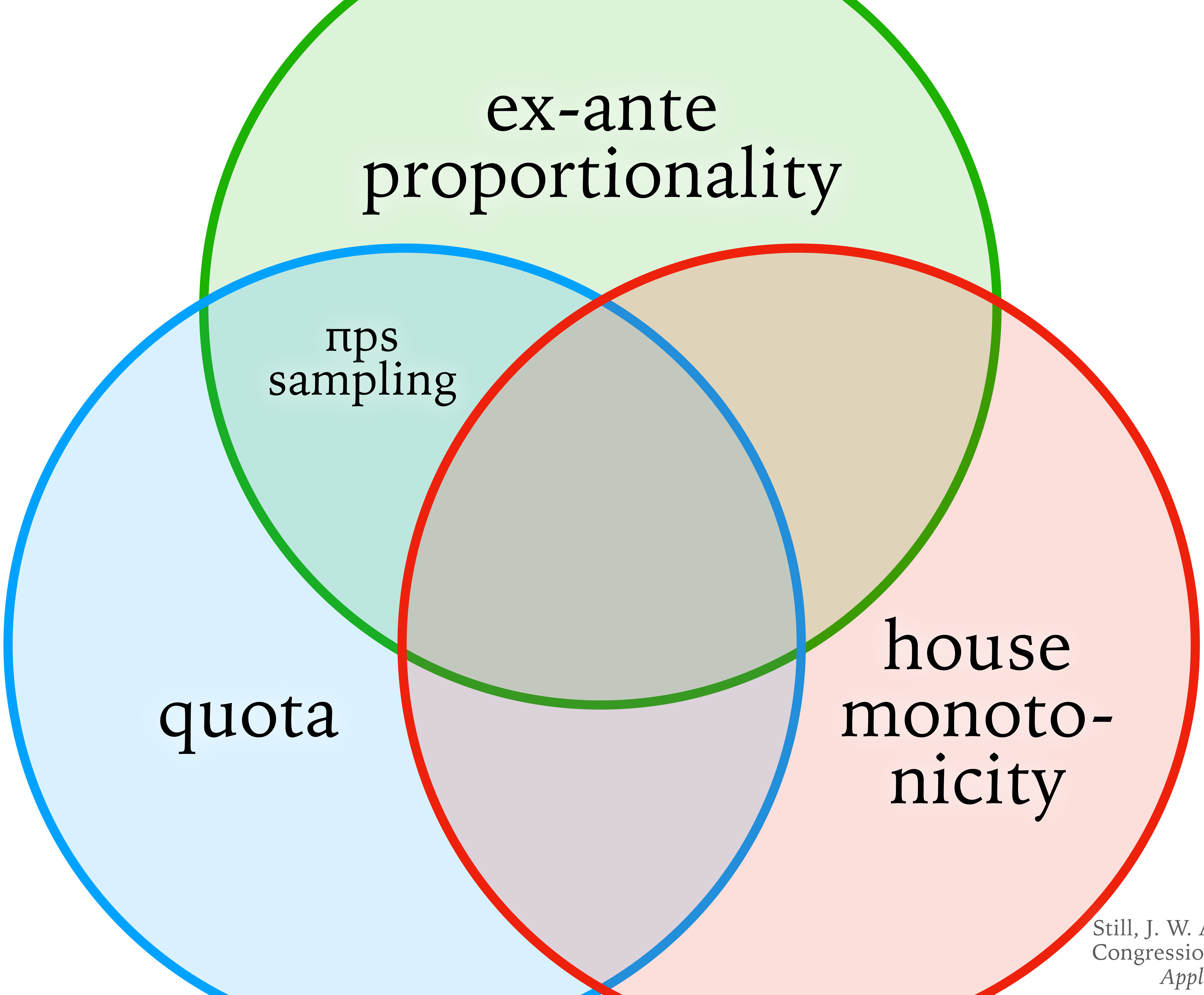


Still, J. W. A Class of New Methods for Congressional Apportionment. *SIAM J. Appl. Math.* 37, 401–418 (1979).



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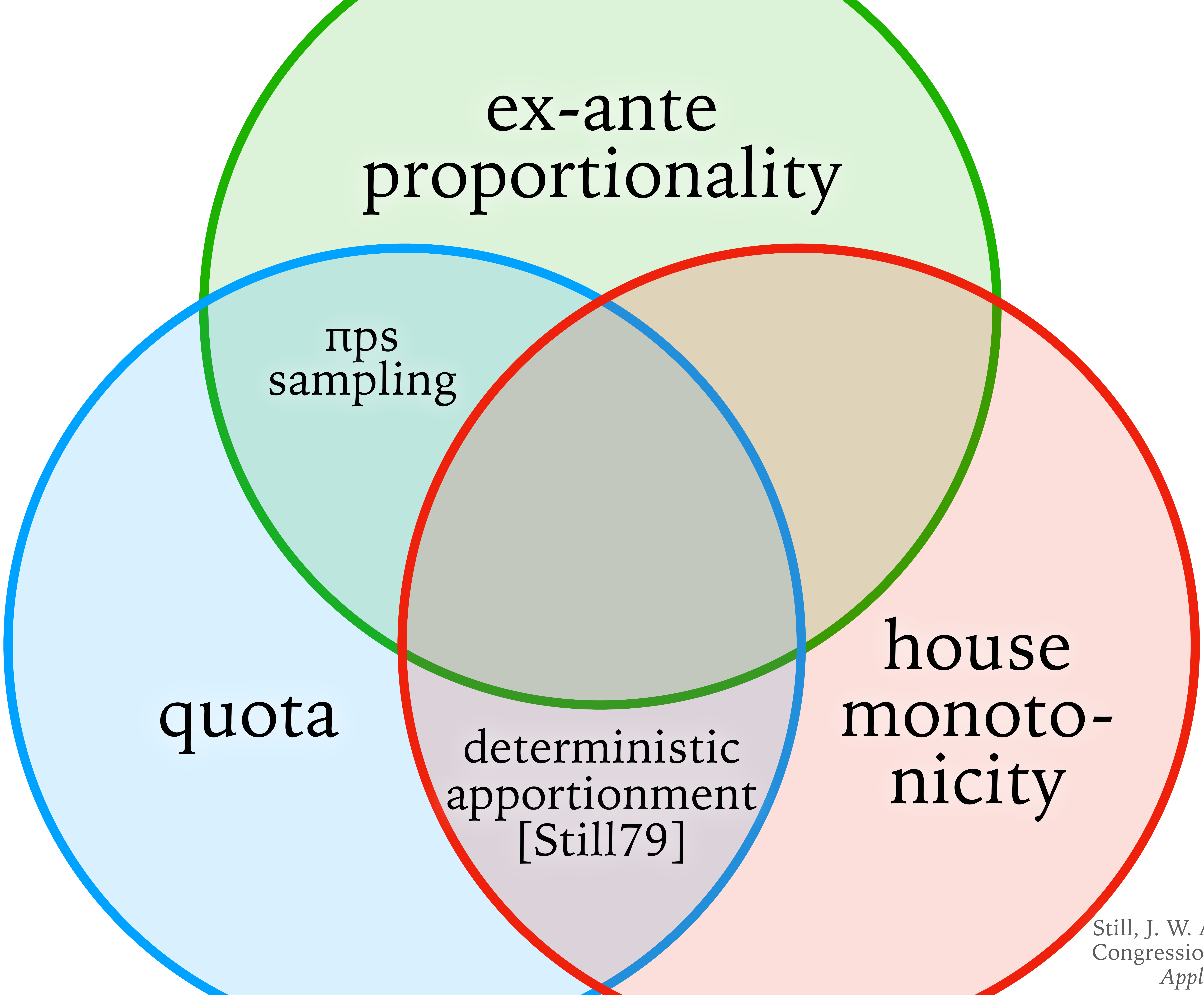

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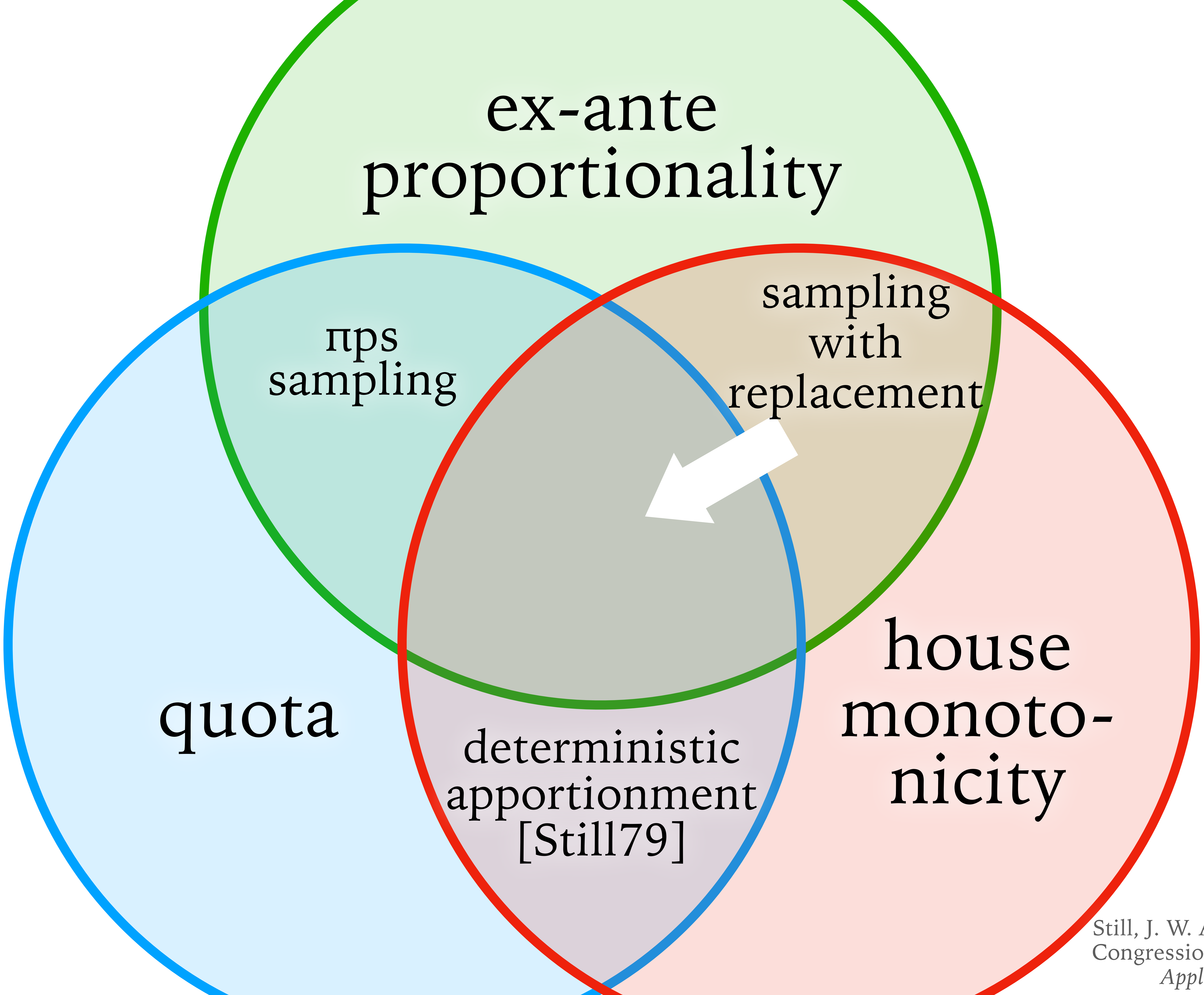

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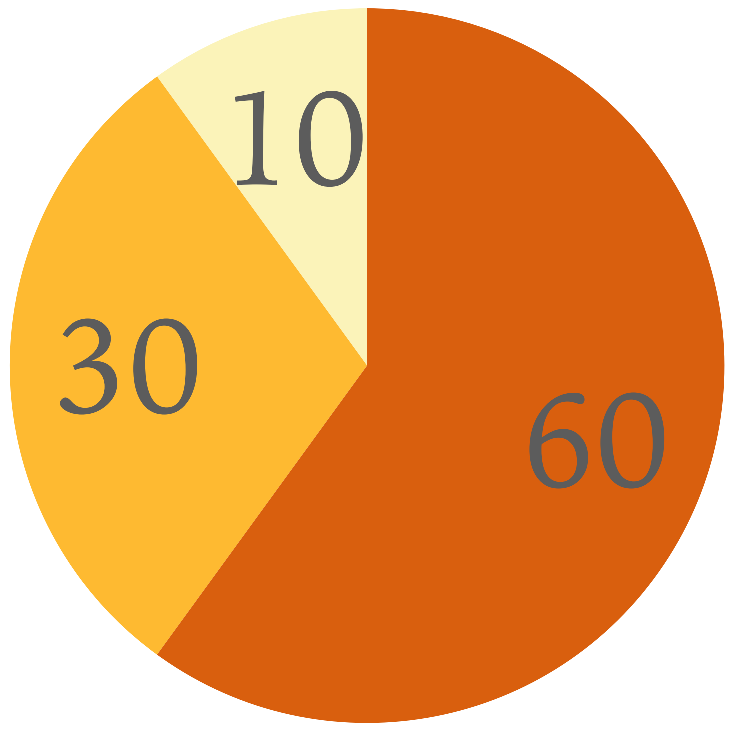
Ex-Ante Proportionality + House Monotonicity



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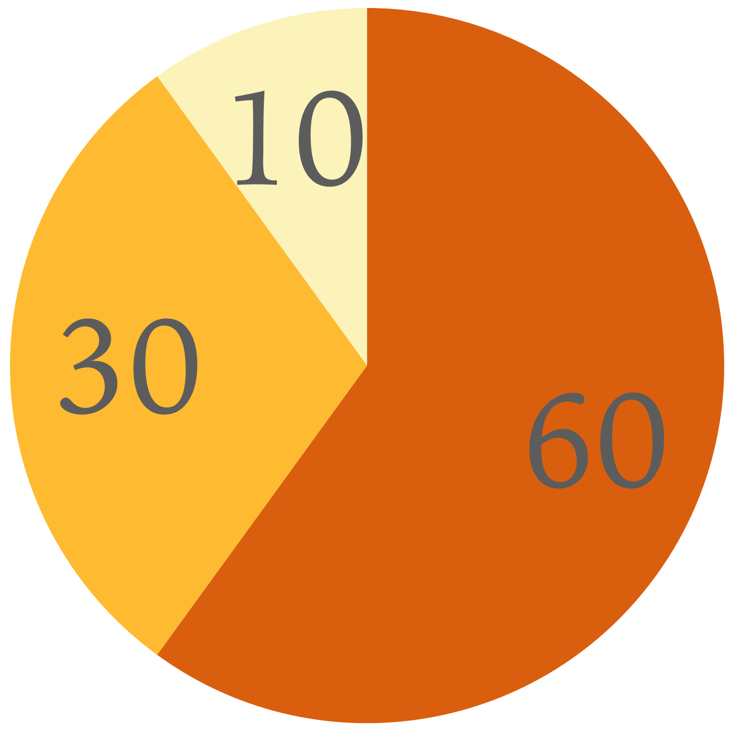
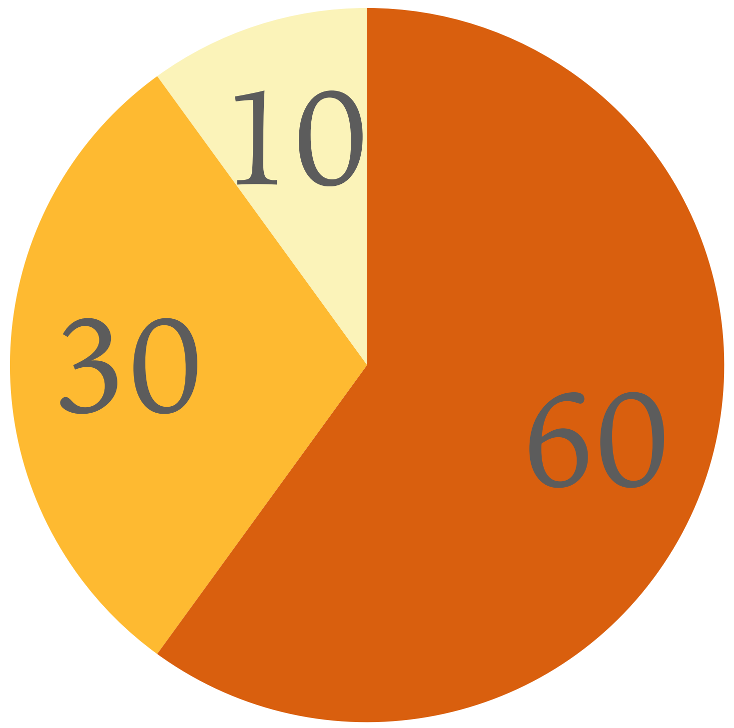


$h = 1$ brown: 60%, orange: 30%, beige: 10%

$h = 2$ brown: 60%, orange: 30%, beige: 10%

$h = 3$ brown: 60%, orange: 30%, beige: 10%

Ex-Ante Proportionality + House Monotonicity



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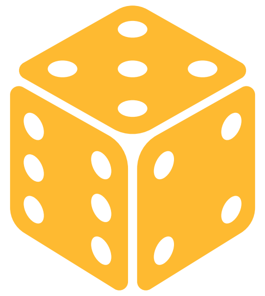
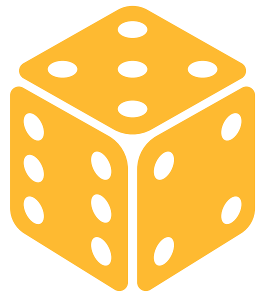
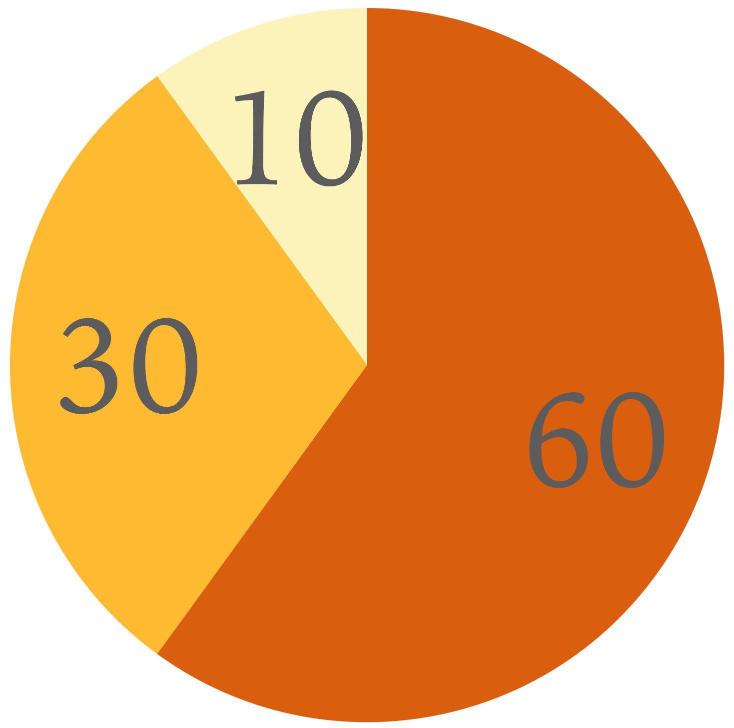


$h = 2$ brown: 60%, orange: 30%, beige: 10%

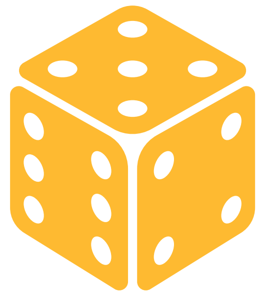


$h = 3$ brown: 60%, orange: 30%, beige: 10%

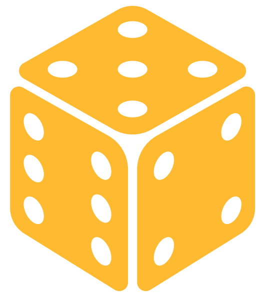
Ex-Ante Proportionality + House Monotonicity



$h = 1$ brown: 60%, orange: 30%, beige: 10% $(0, 1, 0)$



$h = 2$ brown: 60%, orange: 30%, beige: 10% $(0, 2, 0)$



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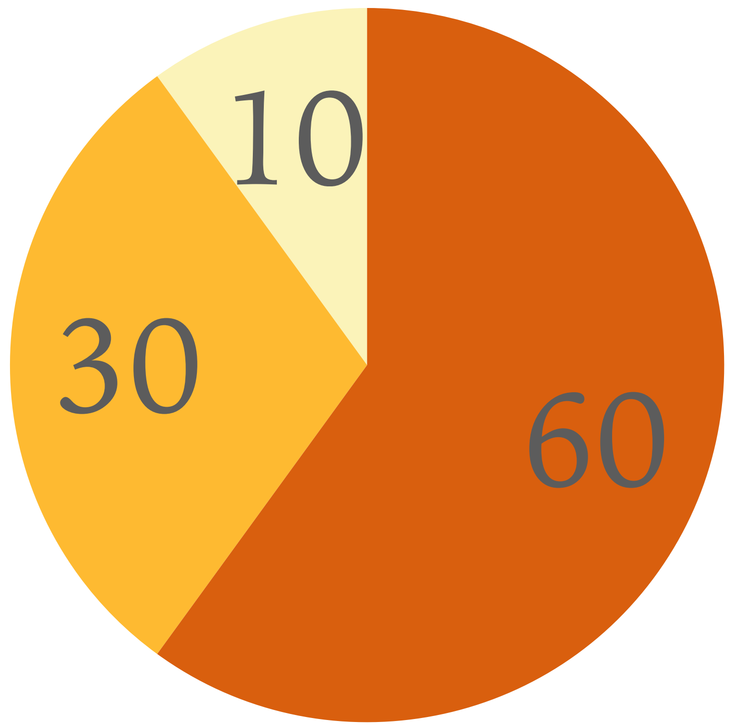

house mono. ✅

$h = 1$ brown: 60%, orange: 30%, beige: 10% $(0, 1, 0)$

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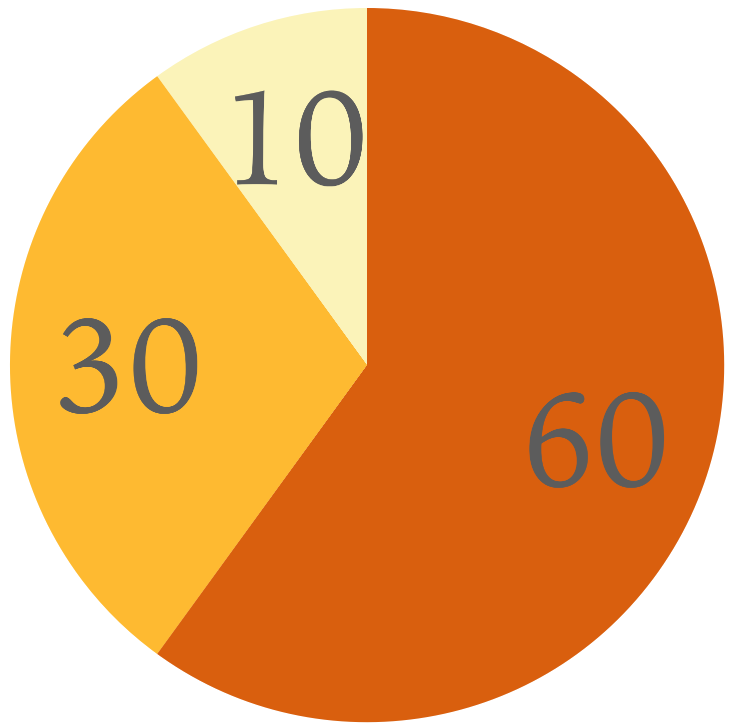

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ex-ante prop. ✅

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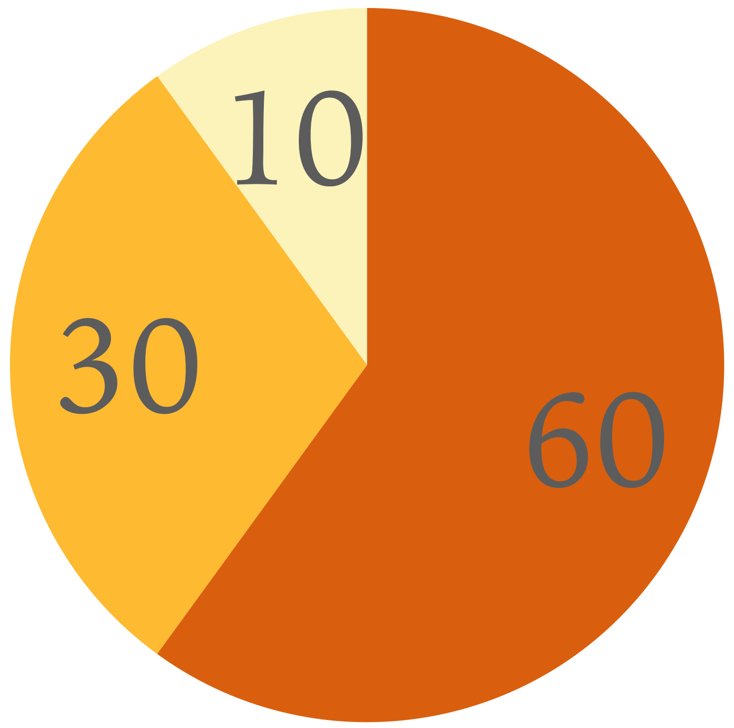

house mono. ✅

ex-ante prop. ✅

quota ❌

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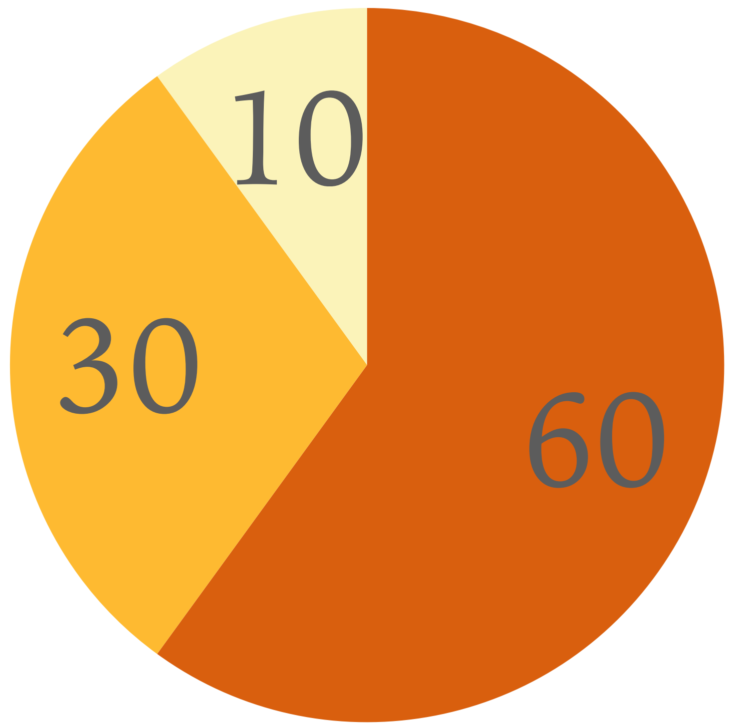

house mono. ✅

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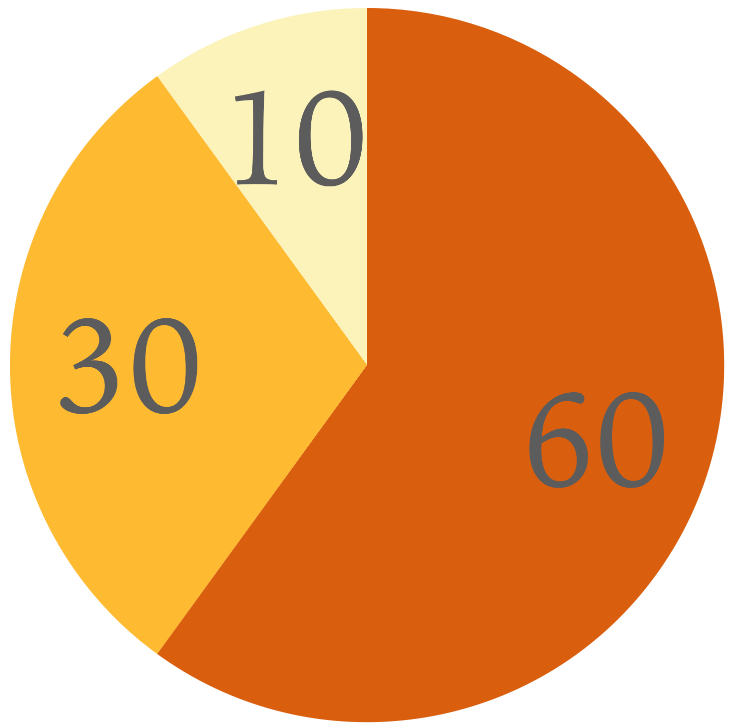
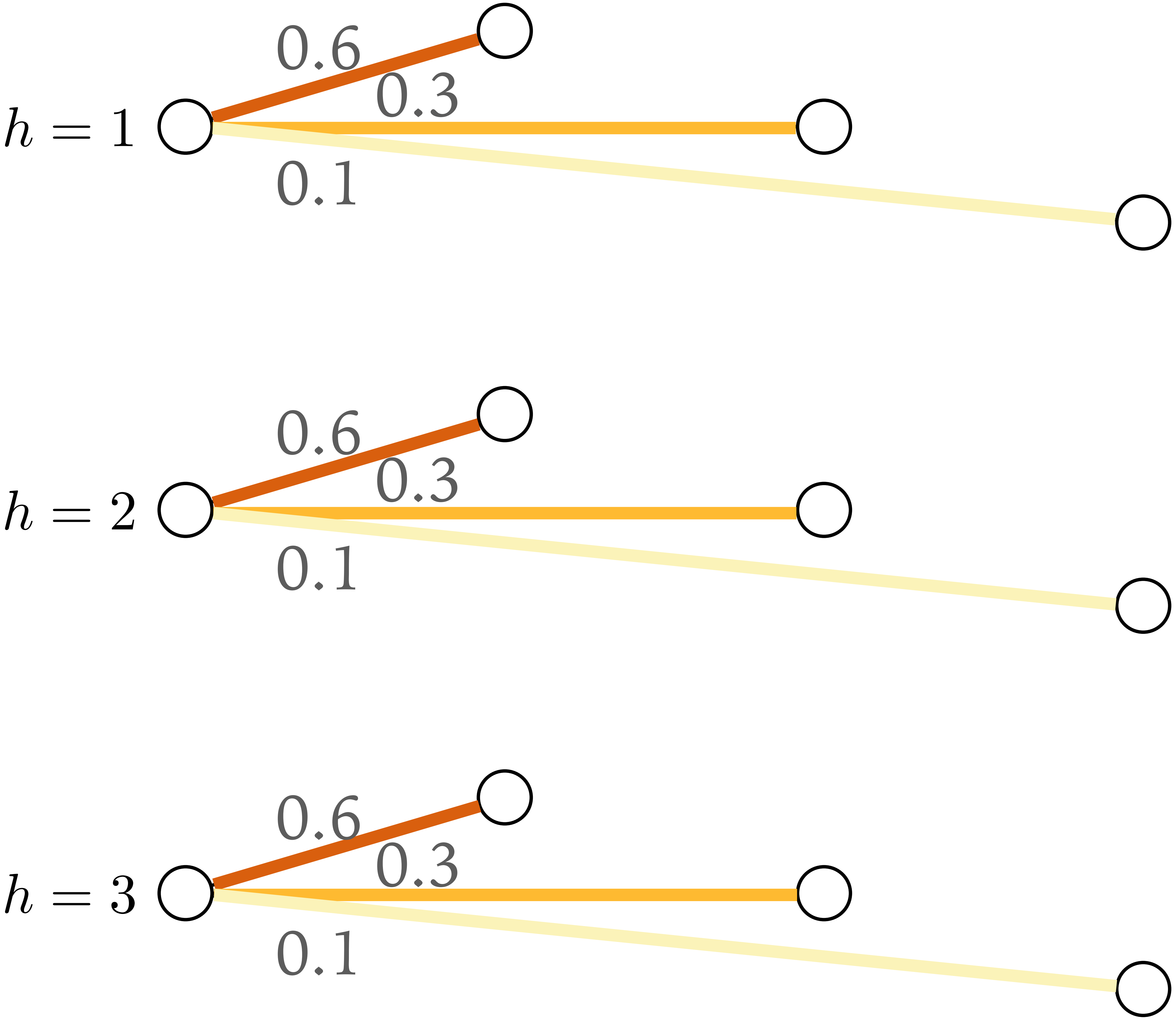
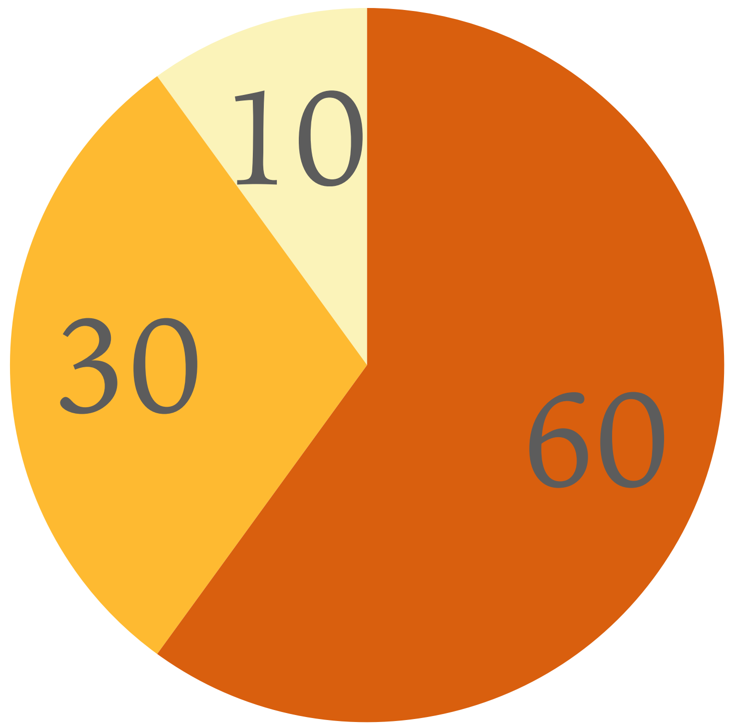
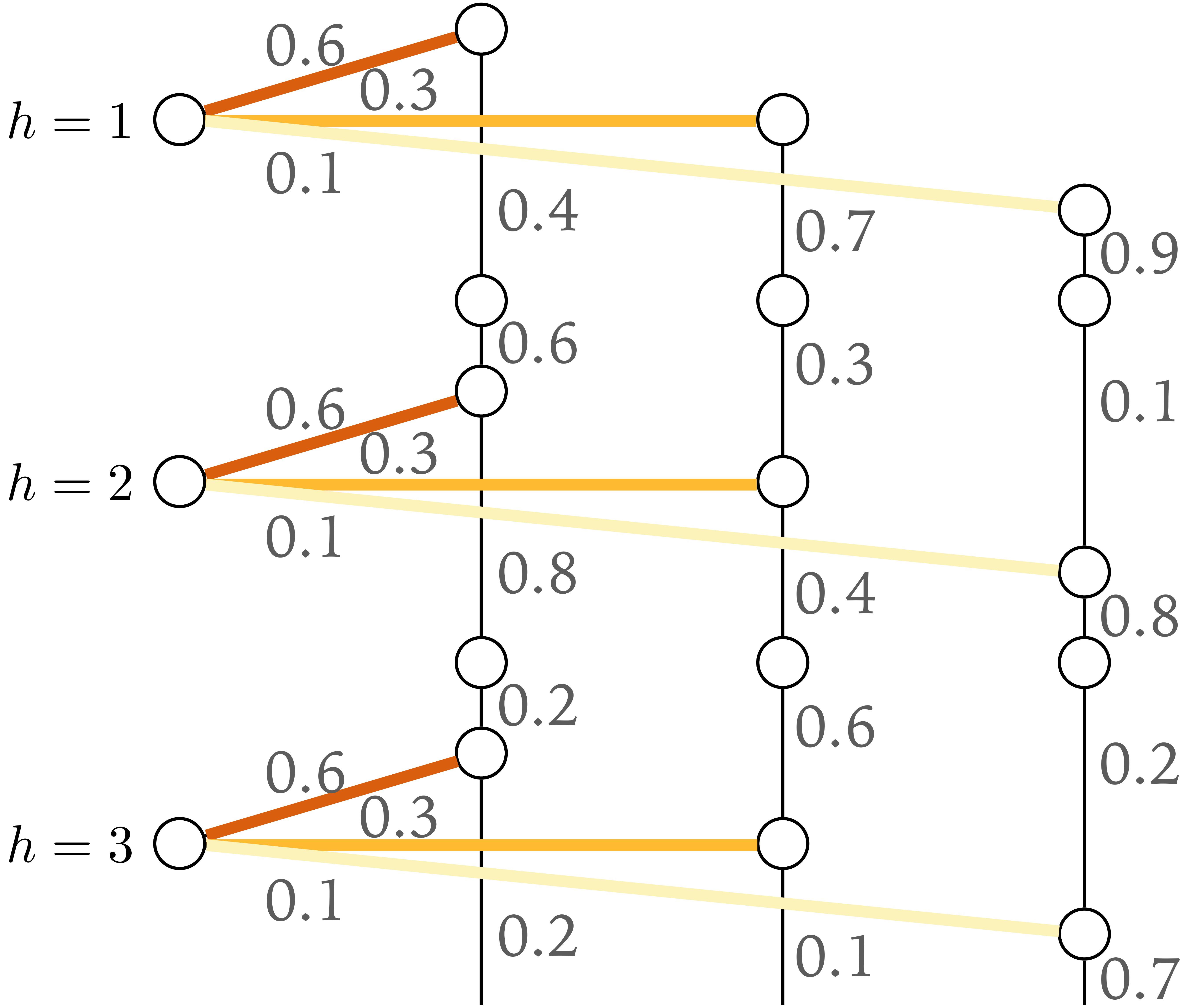
Correlating the Randomness Across Seats





Correlating the Randomness Across Seats





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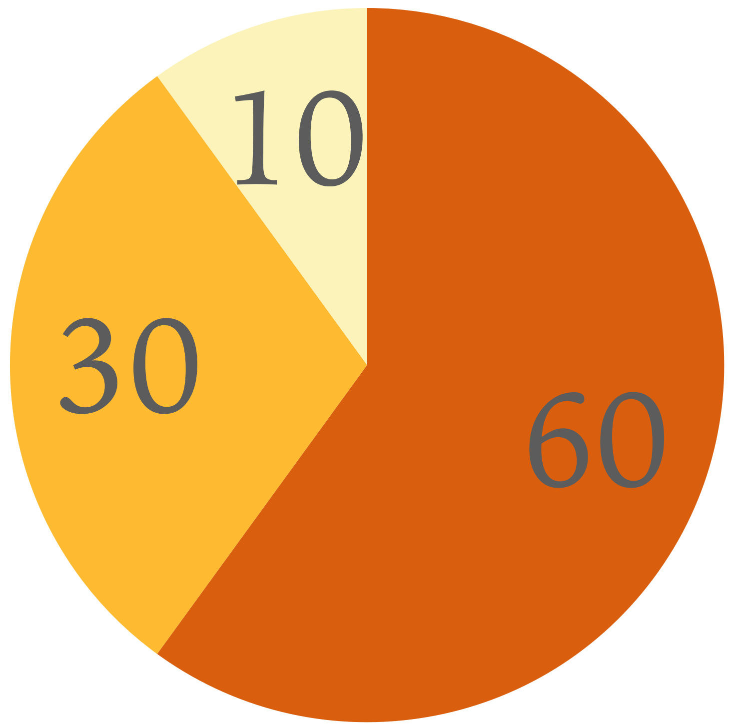

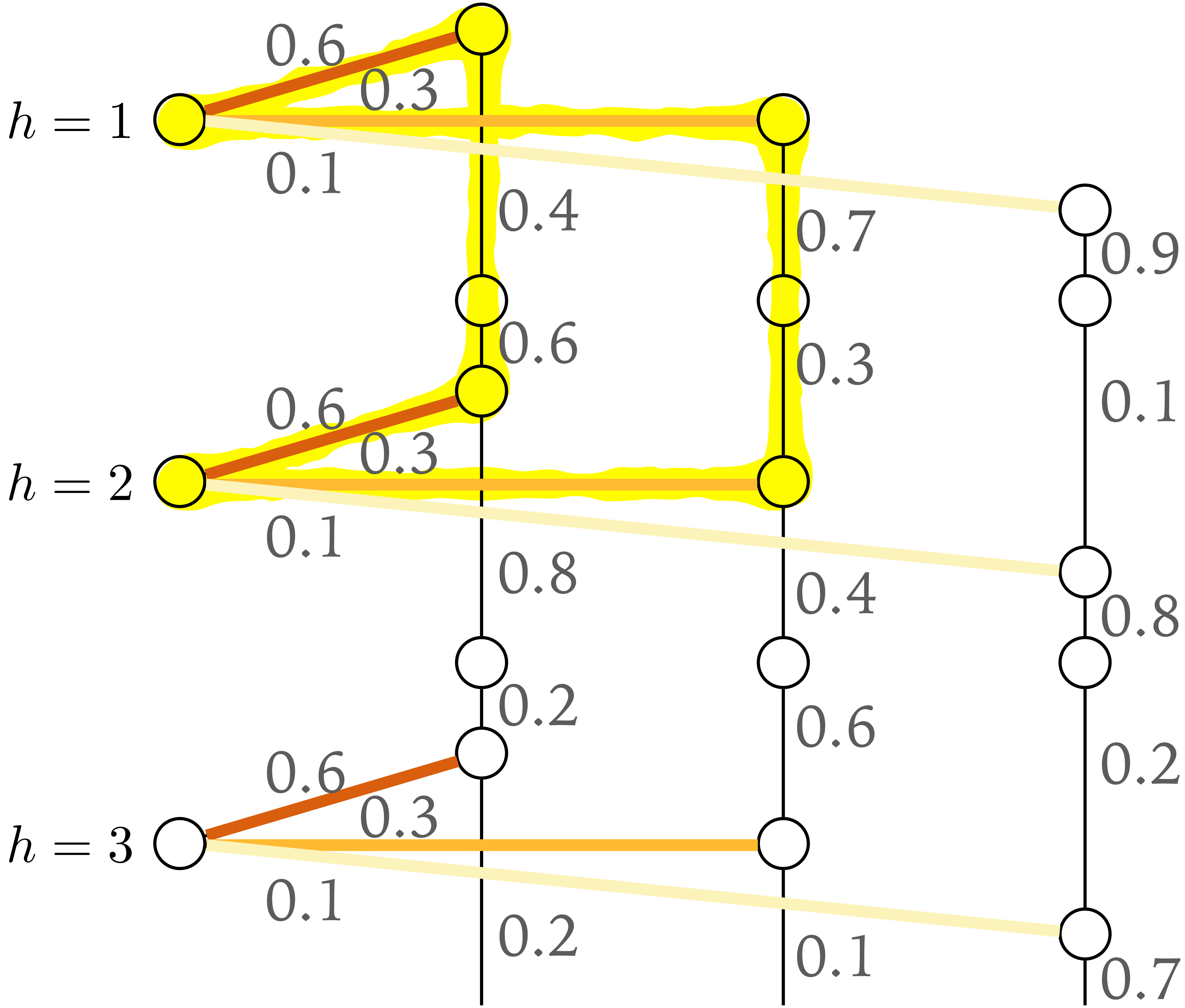

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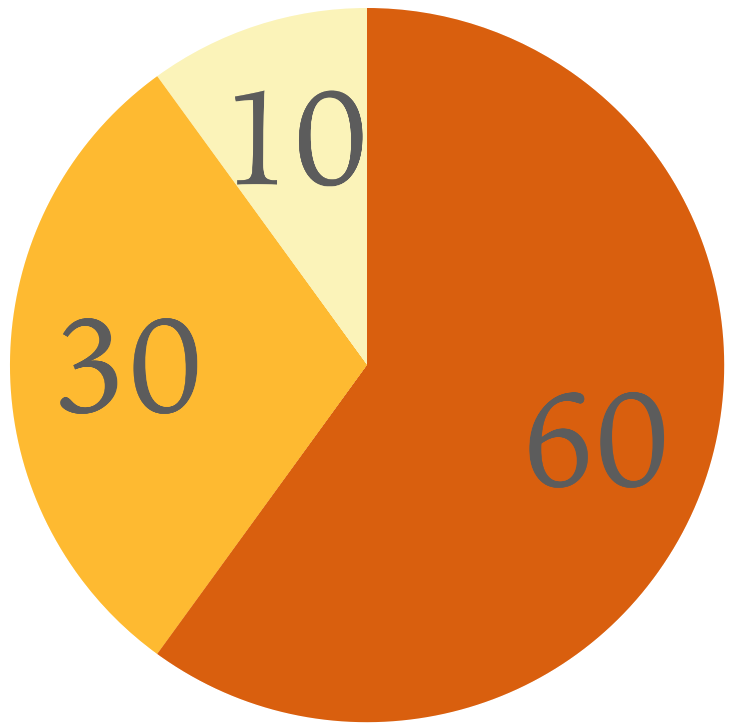

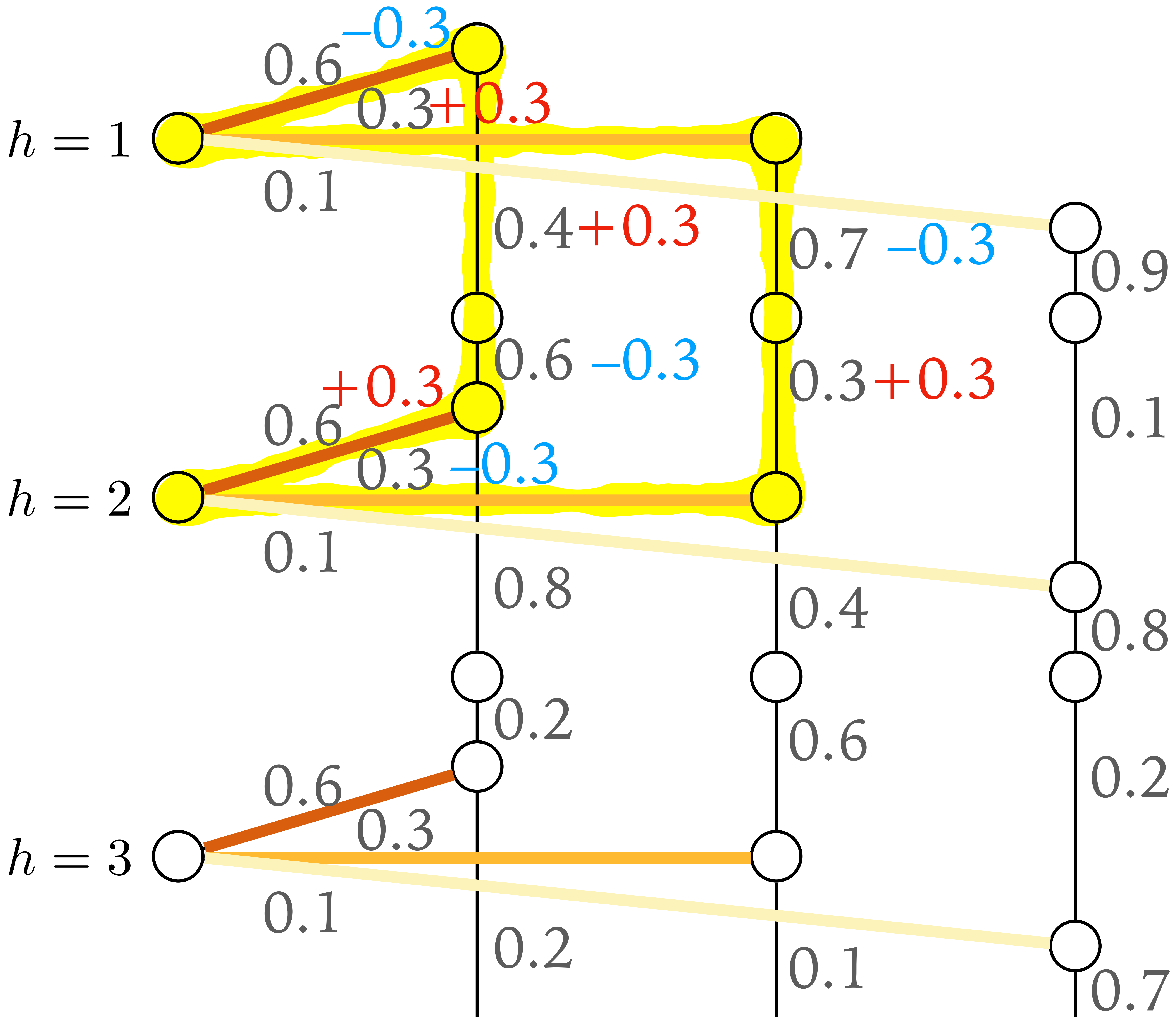

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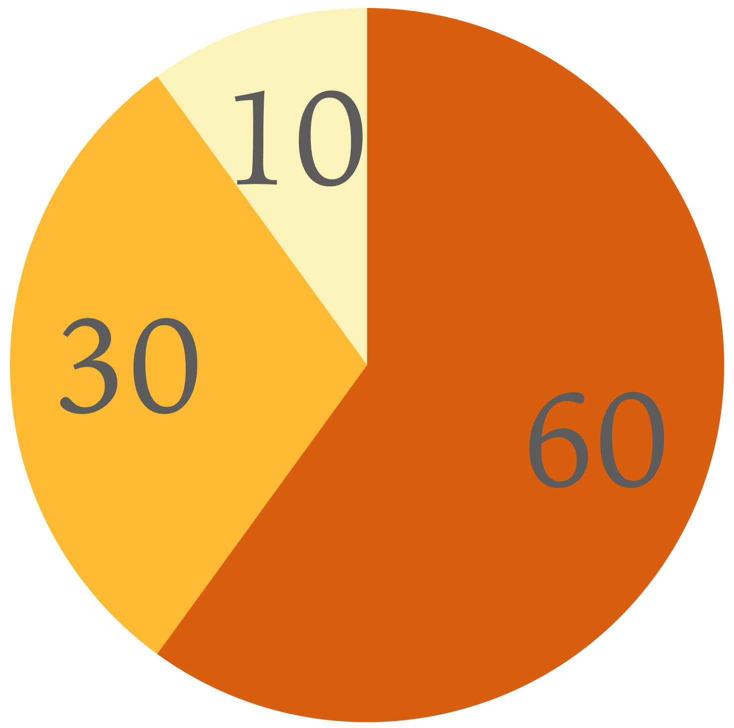

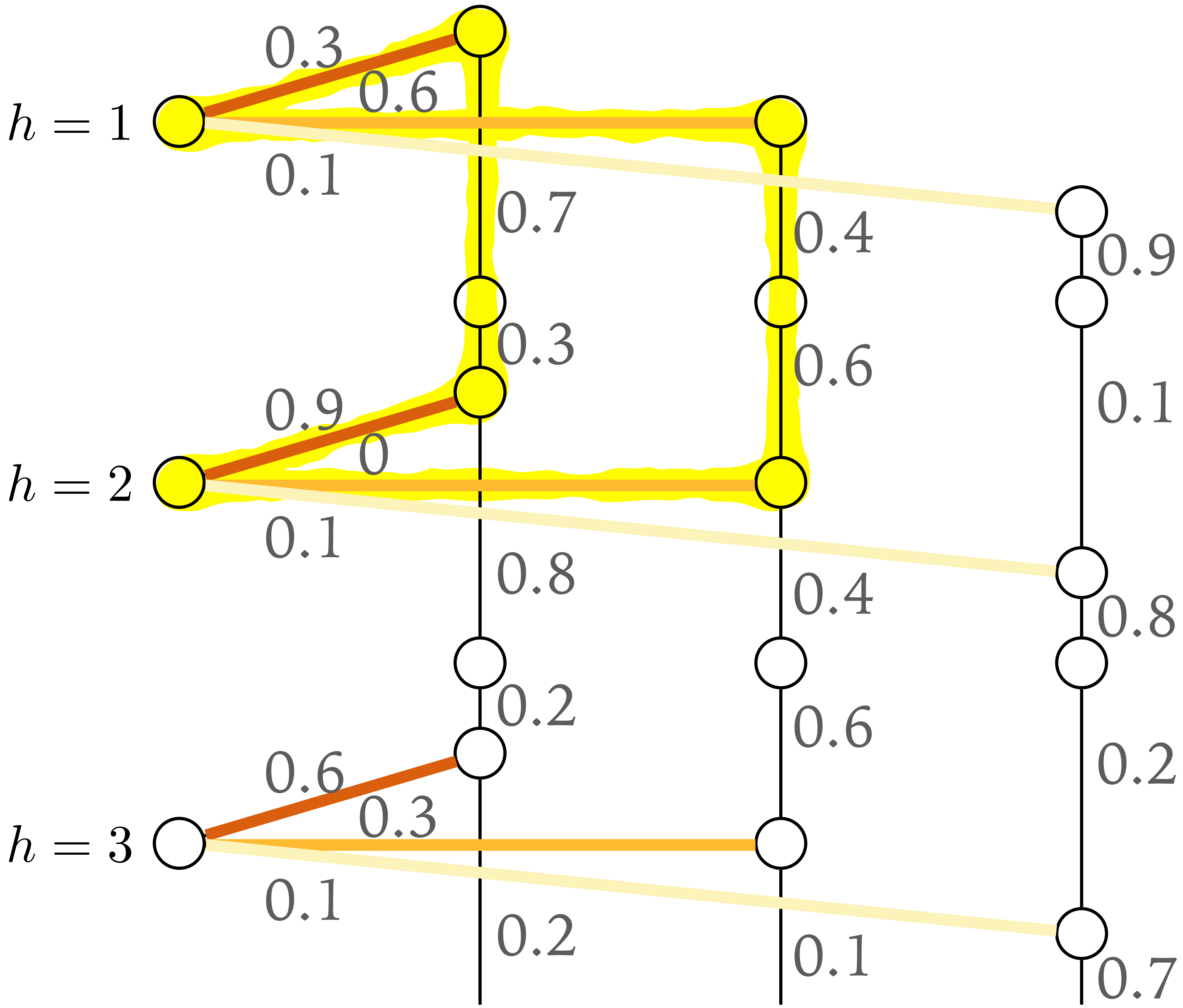

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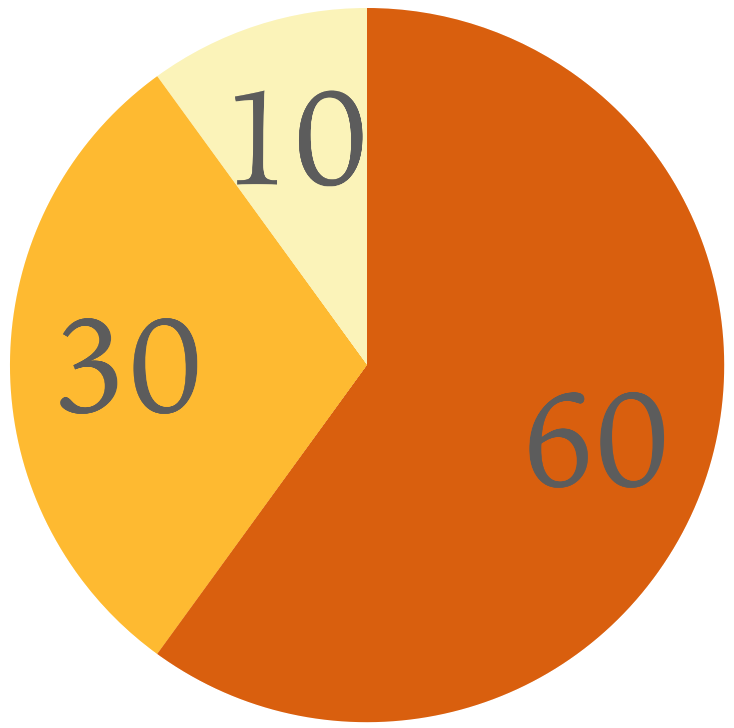
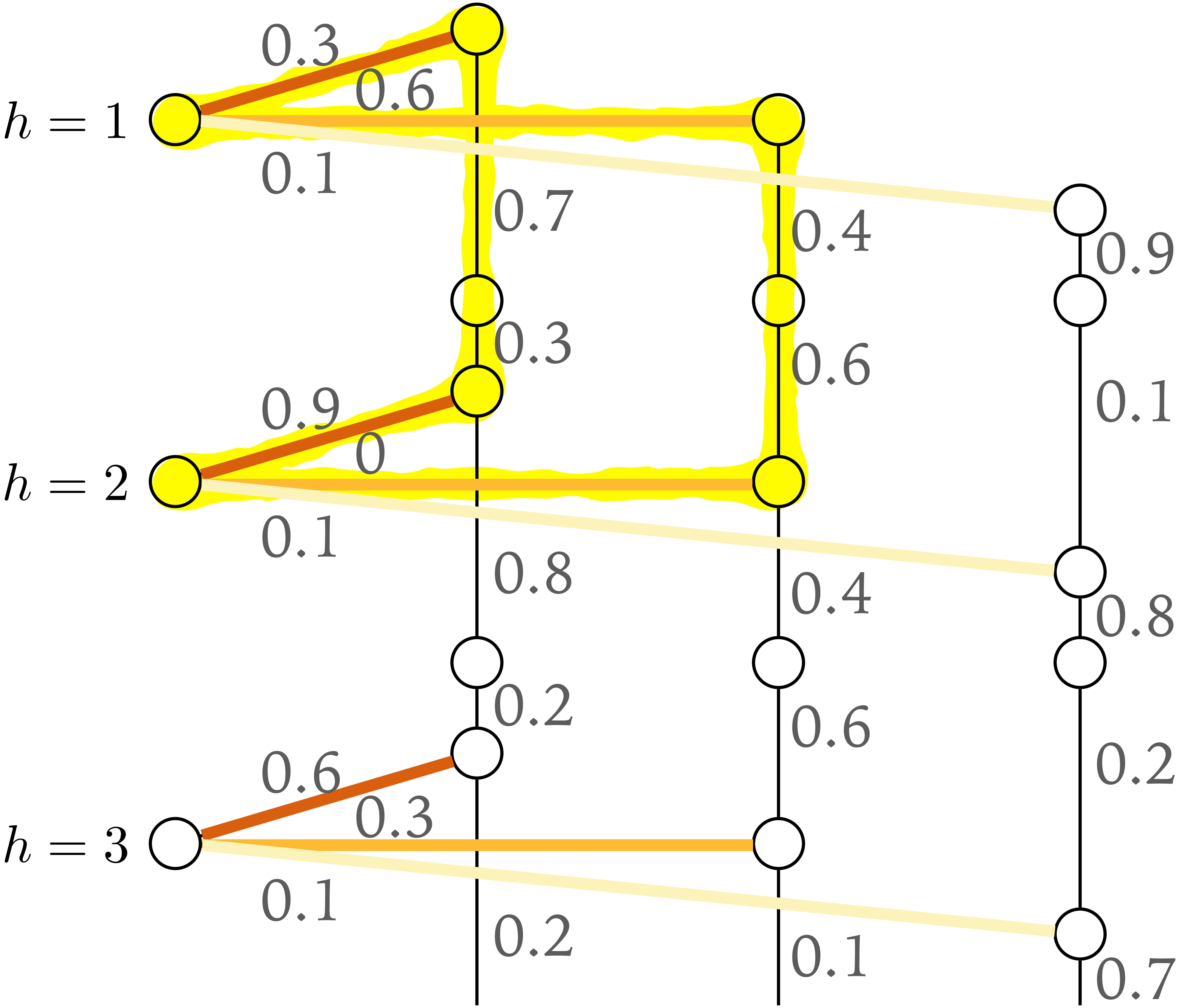


house mono. ✅

ex-ante prop. ✅

quota ✅



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What Else Is This Good for?

Chakraborty, M., Schmidt-Kraepelin, U. & Suksompong, W. Picking sequences and monotonicity in weighted fair division. *Artif. Intell.* 301, 103578 (2021).

What Else Is This Good for?

- Characterization of quota + house monotone solutions as vertices of matching polytope.

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What Else Is This Good for?

- Characterization of quota + house monotone solutions as vertices of matching polytope.
- House-monotone apportionment = picking sequences for weighted fair division. Quota implies WPROP1 [CSS21].

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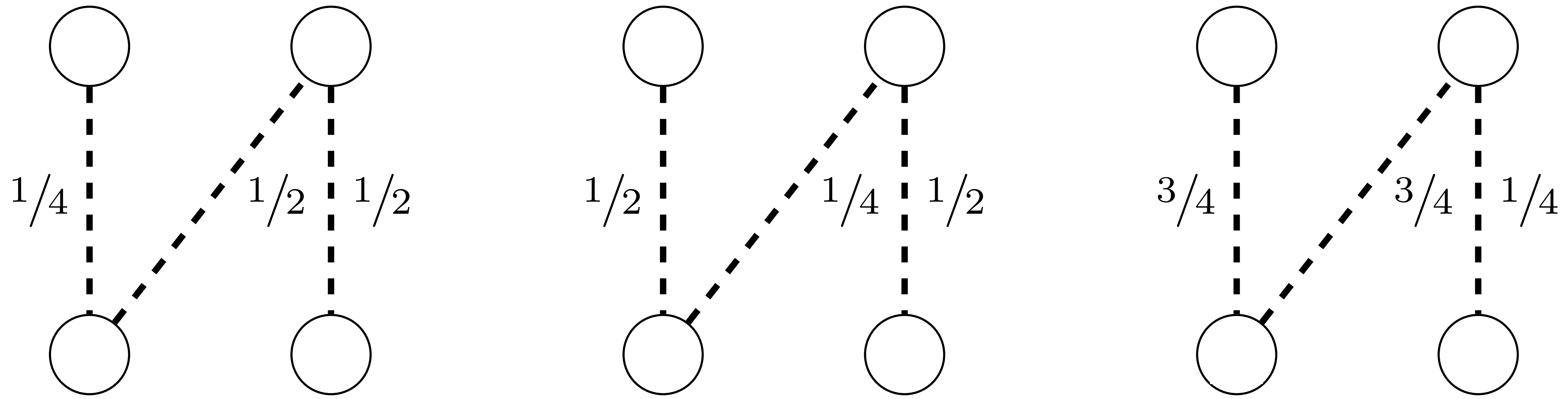
"Quota" in Rounding Bipartite Graphs

Pipage rounding [GKP+06]:



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Pipage rounding [GKP+06]:

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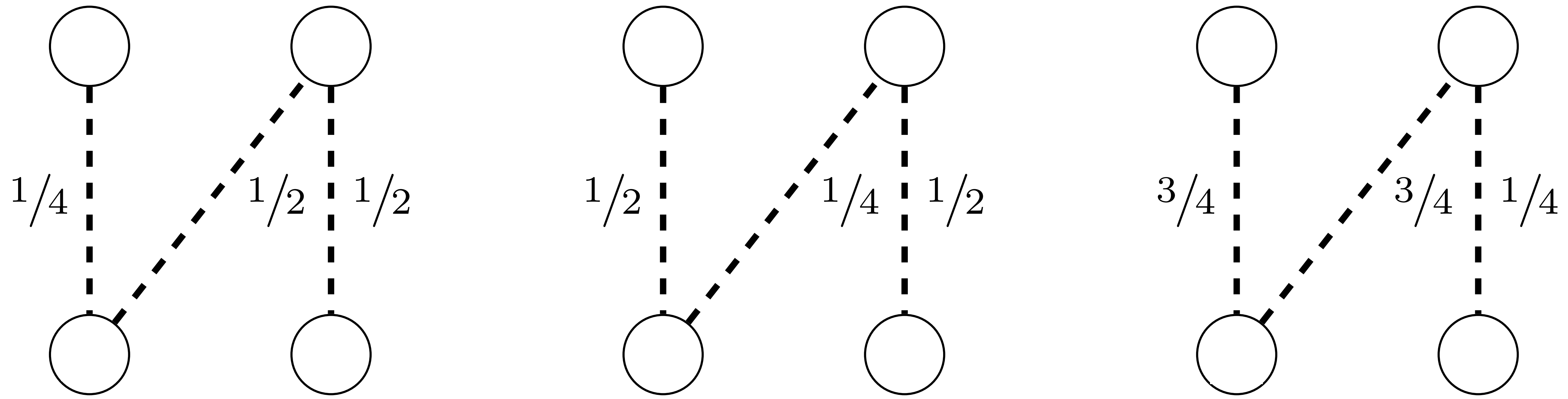

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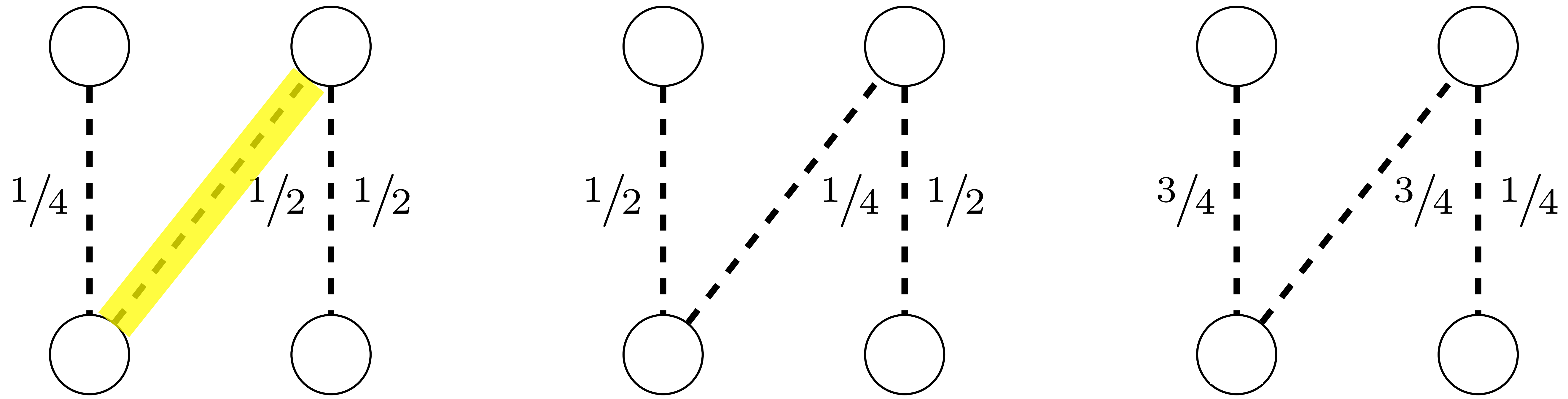

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"Quota" in Rounding Bipartite Graphs

Pipage rounding [GKP+06]:

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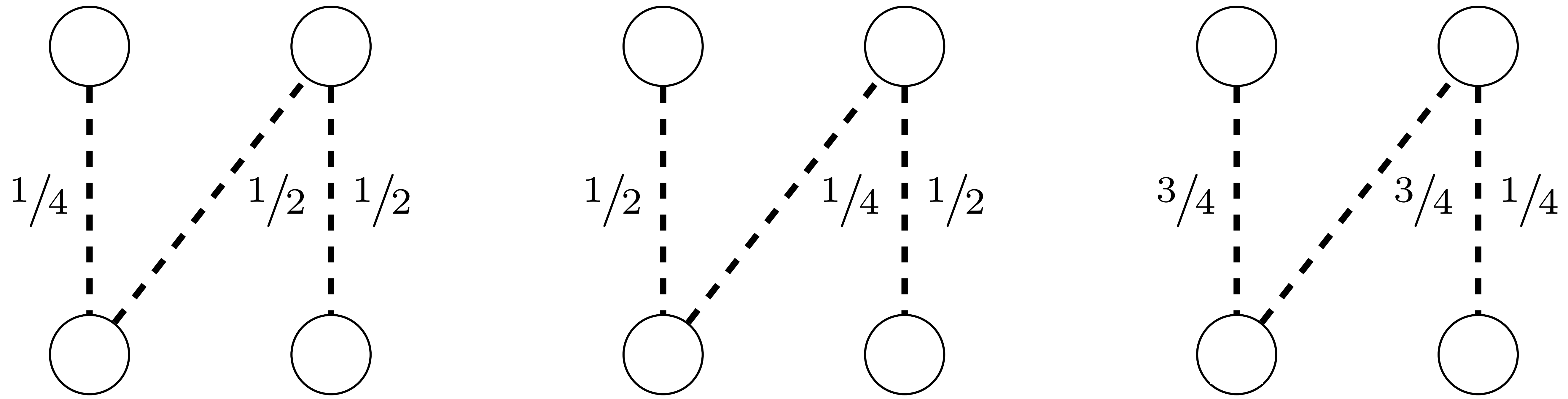
- degree preservation



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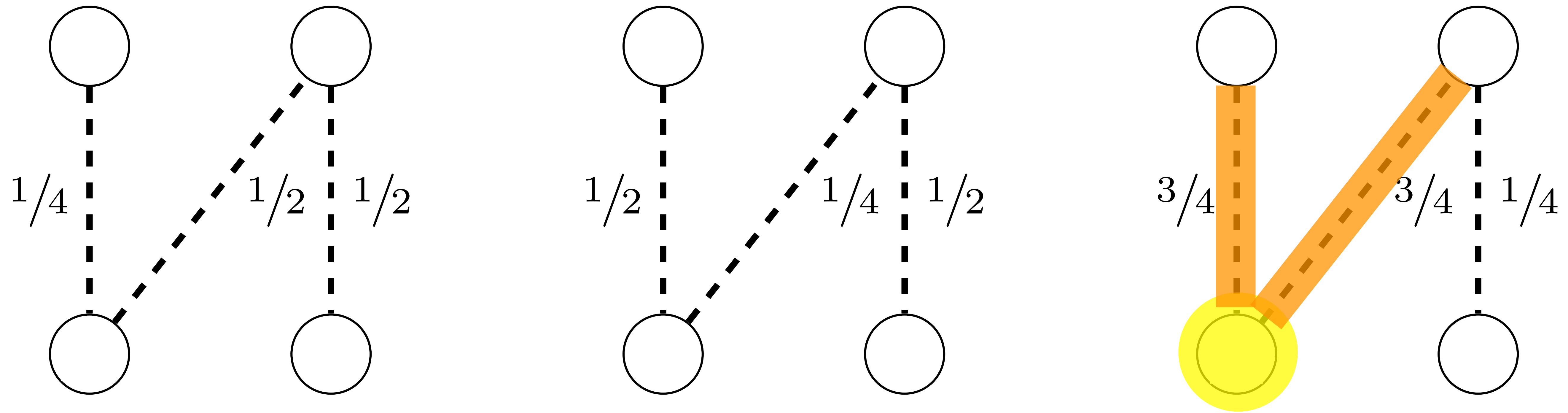

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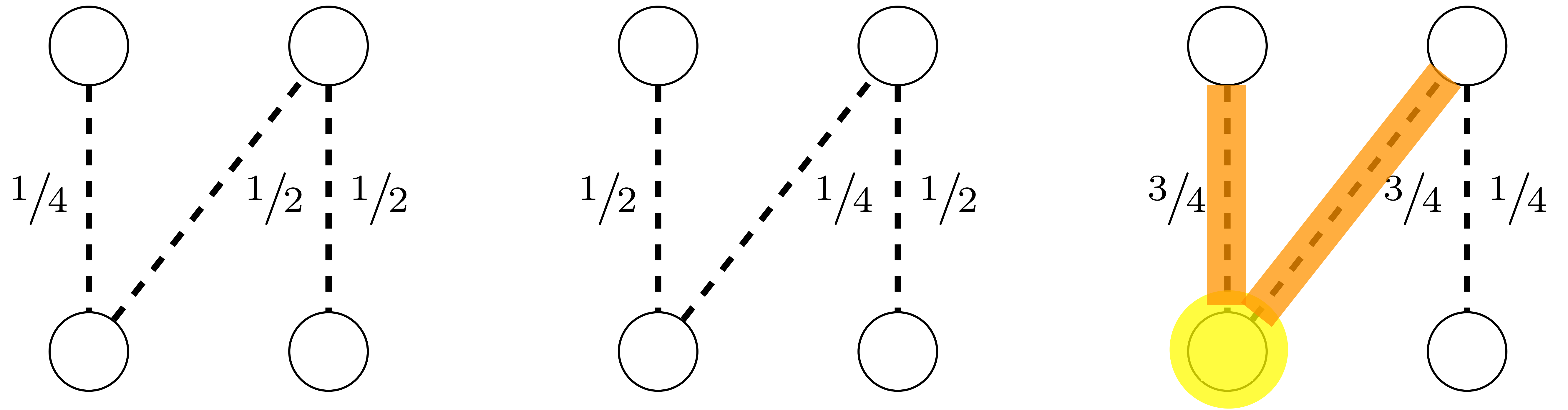

$$\left\lfloor \frac{3}{4} + \frac{3}{4} \right\rfloor \text{or} \left\lceil \frac{3}{4} + \frac{3}{4} \right\rceil$$

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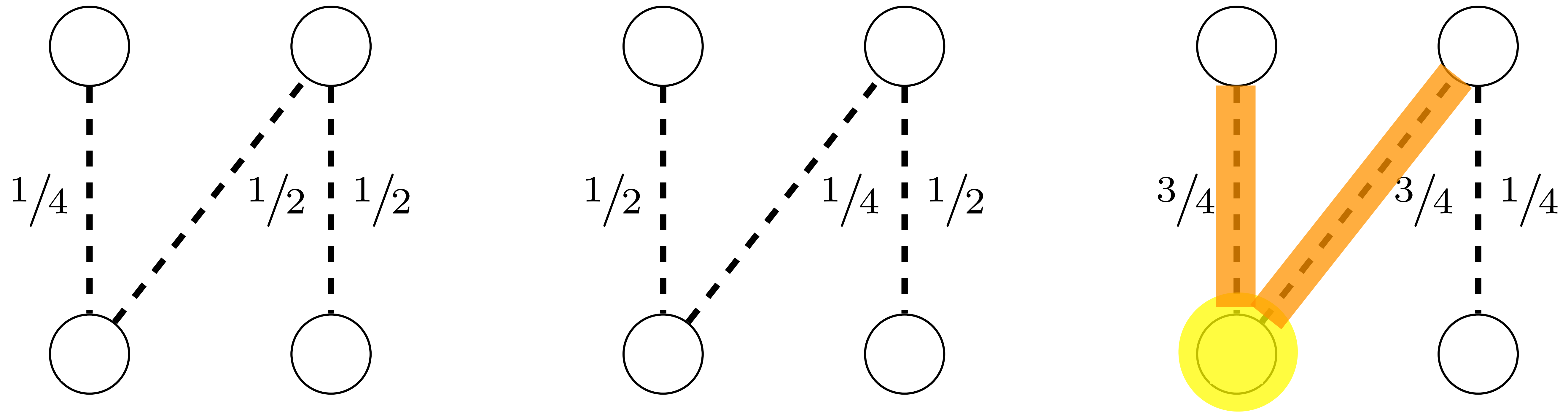
"Quota" in Rounding Bipartite Graphs

Pipage rounding [GKP+06]:

- marginal distribution
- degree preservation
- negative correlation



Gandhi, R. et al. Dependent Rounding and Its Applications to Approximation Algorithms. *J. ACM* 53, 324–360 (2006).

"Quota" in Rounding Bipartite Graphs

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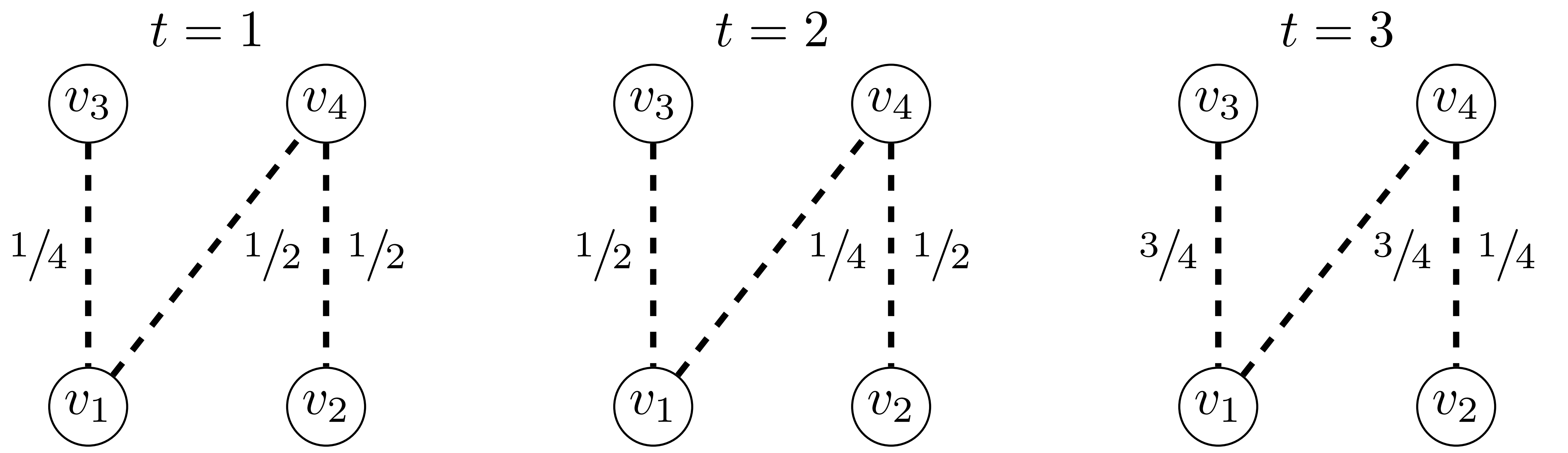

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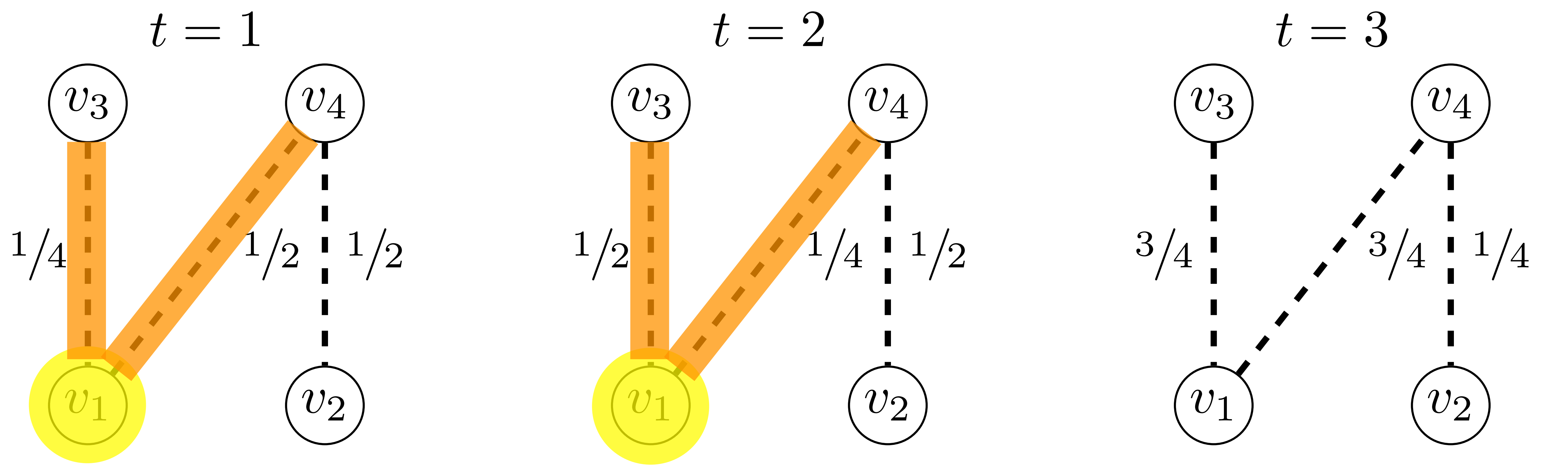

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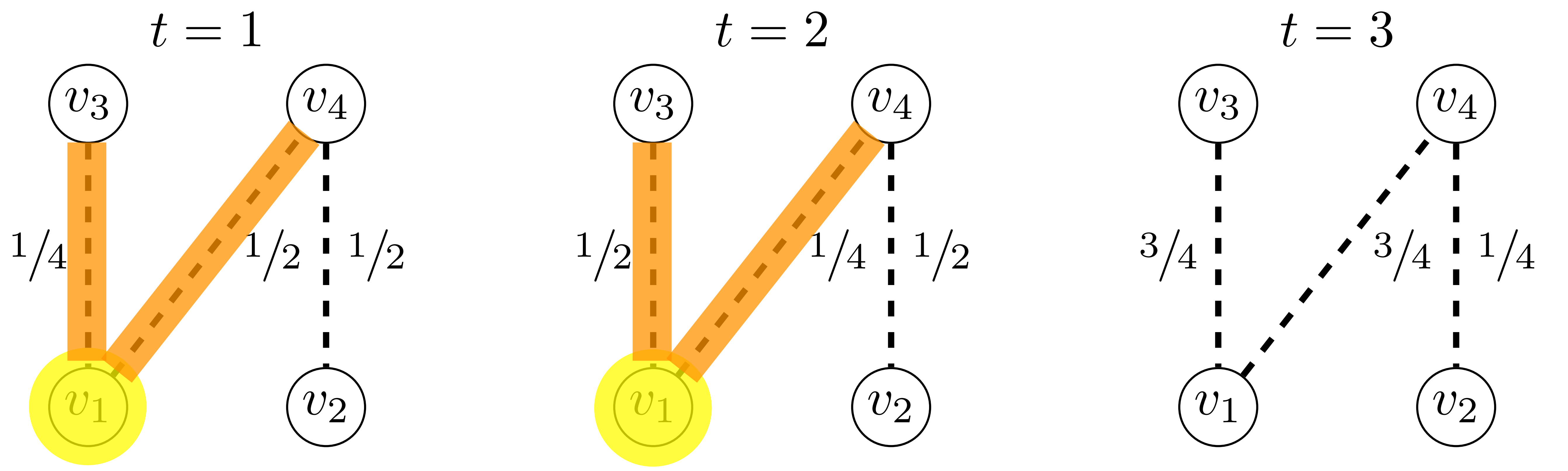

$$\left\lfloor \frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4} \right\rfloor \text{ or } \left\lceil \frac{1}{4}+\frac{1}{2}+\frac{1}{2}+\frac{1}{4} \right\rceil$$

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