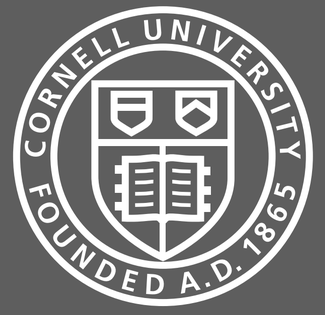
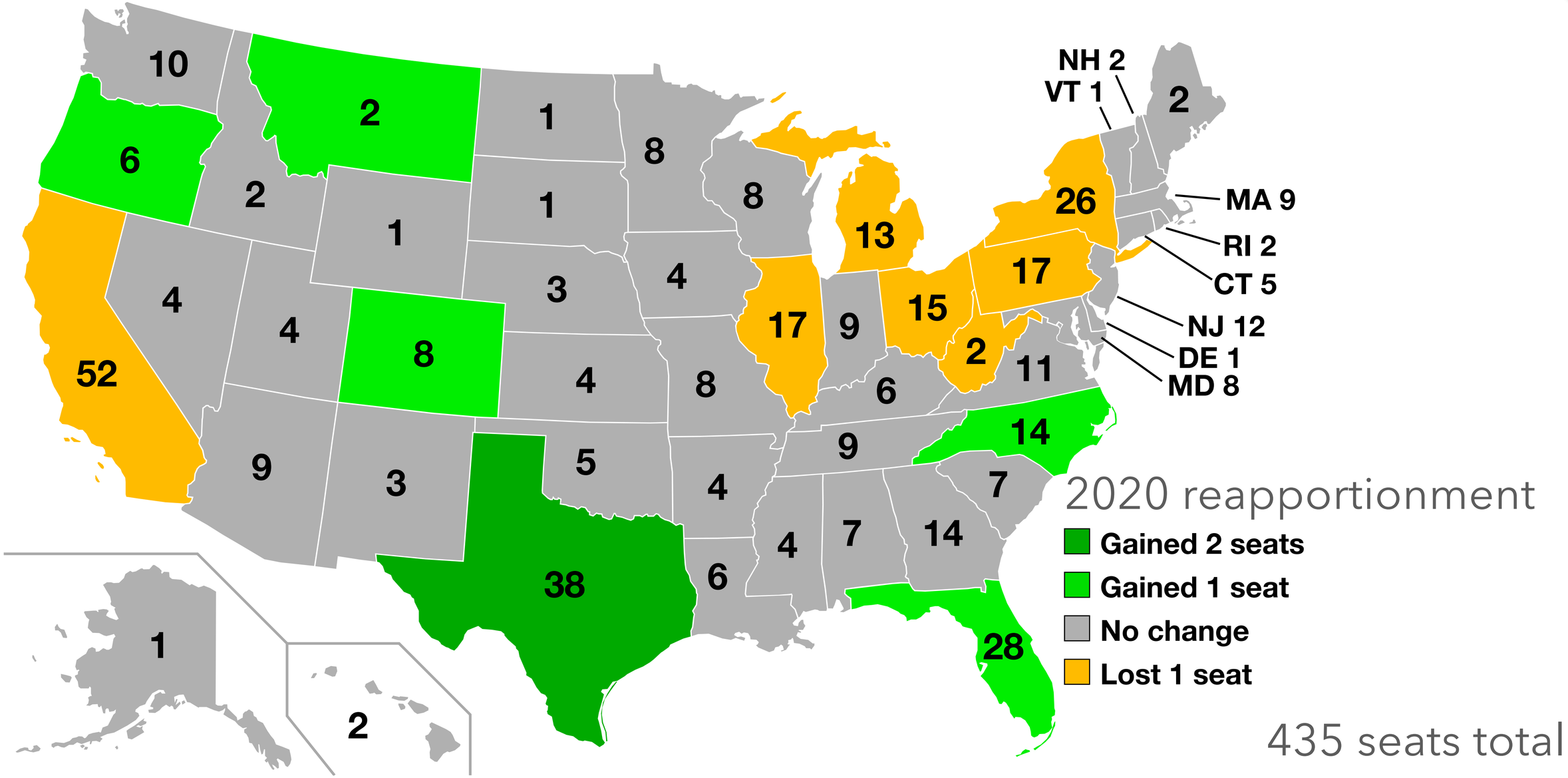
Apportionment in the 19th Century

Paul Gölz

Lecture 20, ORIE 4360/5360, Fall ’26



Splitting House Seats Across States



What the Constitution Says



"Representatives … shall be apportioned among the several States … according to their respective Numbers … The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative."

Can You Be a Founding Father/Mother?

Example:

Three states, population 1,000, 100 seats to allocate.

State	Population	Percentage
Louisibama	506	50.6
Calyoming	307	30.7
Tennemont	187	18.7
Total	1,000	100.0

How many of the 100 seats would you give to each state?

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Model

States $N = \{1, \ldots, n\}$. House size k (i.e., number of seats to be allocated).

State i has a population p_i, total population $P = \sum_{i \in N} p_i$.

Apportionment method is a function f that maps $\boldsymbol{p}, k$ to a vector $(a_1, \ldots, a_n)$ of nonnegative integers such that $\sum_{i \in N} a_i = k$.

State i's standard quota is $q_i := p_i / P \cdot k$.

Proportionality

Ideally, want $a_i = q_i$, state i's proportional share of seats.

q_i is not an integer! Need to round somehow.

State	Population	Percentage	q_i	round(q_i)
Louisibama	506	50.6	50.6	51
Calyoming	307	30.7	30.7	31
Tennemont	187	18.7	18.7	19
Total	1,000	100.0	100.0	101

Arithmetic rounding might not allocate k seats.

First Contender: Alexander Hamilton



(1757–1804)

First treasury secretary, an author of the Federalist Papers, and star of famous 2015 musical.

Hamilton's Method

Call $\lfloor q_i \rfloor$ the lower quota; $\lceil q_i \rceil$ the upper quota. $q_i - \lfloor q_i \rfloor$ the residue.

Hamilton's method allocates each state its lower quota. Then, allocates remaining seats one at a time to states with largest residue.

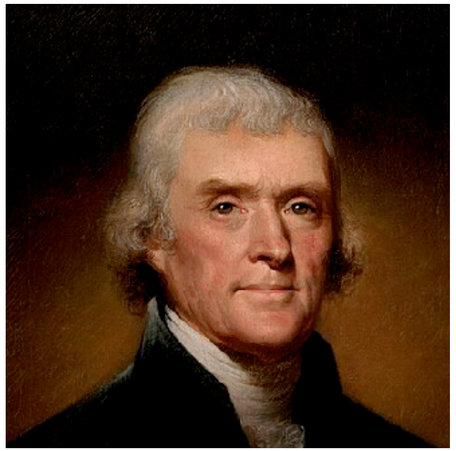
1792 bill passed by Congress would have apportioned based on Hamilton's method.

Hamilton's Method in 1792

1790 census. 120 seats to be allocated, 30k citizens per representative.

State	Population	Population / 30k	Floor	Final seats
Connecticut	236,841	7.895	7	**8**
Delaware	55,540	1.851	1	**2**
Georgia	70,835	2.361	2	2
Kentucky	68,705	2.290	2	2
Maryland	278,514	9.284	9	9
Massachusetts	475,327	15.844	15	**16**
New Hampshire	141,822	4.727	4	**5**
New Jersey	179,570	5.986	5	**6**
New York	331,589	11.053	11	11
North Carolina	353,523	11.784	11	**12**
Pennsylvania	432,879	14.419	14	14
Rhode Island	68,446	2.282	2	2
South Carolina	206,236	6.875	6	**7**
Vermont	85,533	2.851	2	**3**
Virginia	630,560	21.019	21	21
Total	3,615,920	120.531	112	120

Thomas Jefferson



(1743–1826)

In 1792, Secretary of State. Became 2nd Vice-President and 3rd President.

Jefferson's Method

Finds a divisor $D \in \mathbb{R}_{>0}$ such that $\sum_{i \in N} \lfloor p_i / D \rfloor = k$.

Then, state i receives $\lfloor p_i / D \rfloor$ seats.

After George Washington vetoed bill with Hamilton's method in 1792, Congress adopted Jefferson's method. Used until 1830.

Example

$k = 100$

		$D = 10$		$D = 9.8$		$D = 9.9$	
State	p_i	p_i/D	a_i	p_i/D	a_i	p_i/D	a_i
Louisibama	506	50.6	50	51.6	51	51.1	51
Calyoming	307	30.7	30	31.3	31	31.0	31
Tennemont	187	18.7	18	19.1	19	18.9	18
Total	1,000		98		101		100

Jefferson's Large-State Bias

$k = 100$

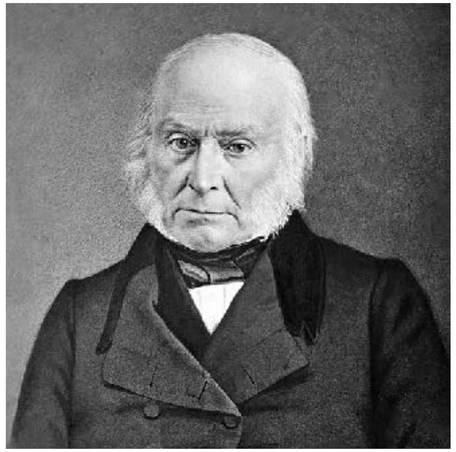
	Population	*Divisor 100,000*	*Seats*	*Divisor 97,000*	*Seats*	*Ratio*
Neware	2,620,000	26.20	26	27.01	27	97,037
Delayork	168,000	1.68	1	1.73	1	168,000
.		.		.		
.		.		.		
.		.	72	.	72	
.		.		.		
Total	10,000,000		99		100	100,000

Neware gets additional seat despite initially having smaller residue.

When the divisor is reduced, each seat requires 3,000 fewer citizens, and Neware gains for each of its 26 seats.

Neware needs 97,037 citizens per seat whereas Delayork needs 168,000.

John Quincy Adams



(1767–1848)

Sixth US president of US, son of the second US president.

Known for his stance against slavery.

Adams' Method

Finds a divisor $D \in \mathbb{R}_{>0}$ such that $\sum_{i \in N} \lceil p_i/D \rceil = k.$

Then, state i receives $\lceil p_i/D \rceil$ seats.

Large states opposed this proposal, was never adopted.

Adams' Small-State Bias

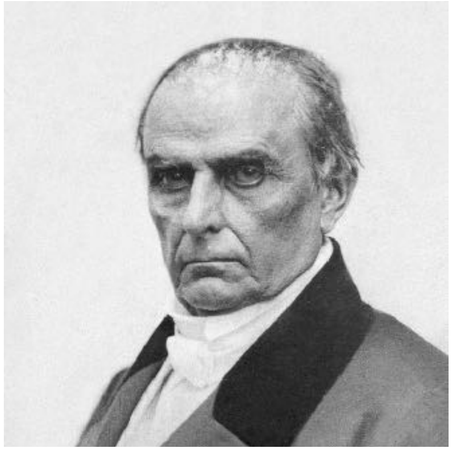
	Population	*Divisor 100,000*	*Seats*	*Divisor 104,000*	*Seats*	*Ratio*
Neware	2,668,000	26.68	27	25.65	26	102,615
Delayork	120,000	1.20	2	1.15	2	60,000
.		.		.		
.		.		.		
.		.	72	.	72	
.		.		.		
Total	10,000,000		101		100	100,000

Neware lost a seat despite initially having larger residue.

When the divisor is increased, each seat requires 4,000 more citizens, and Neware loses for each of its 27 seats.

Neware needs 102,615 citizens per seat whereas Delayork needs 60,000.

Daniel Webster



(1782–1852)

Senator from Massachusetts.

Known for oratory talents, working on his speeches while fishing.

Webster's Method

Write $[x]$ for the arithmetic rounding of x (e.g., $[3.5] = 4$, $[3.4] = 3$).

Adam's method finds a divisor $D \in \mathbb{R}_{>0}$ such that $\sum_{i \in N} [p_i/D] = k$.

Then, state i receives $[p_i/D]$ seats.

Adopted by Congress in 1842.

Question

4 states,
population 200,000,
$k = 20$ seats.

State	Population
A	13,000
B	14,000
C	63,000
D	110,000
Total	200,000

How many seats does state C get for Webster's apportionment?

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Hamilton's After All!



In 1850, Senator Samuel Vinton proposed an apportionment method that is identical to Hamilton's.

"Vinton's" method was adopted that year, and house size was increased to 234, for which Webster's and Hamilton's method agree.

House grew to 241 in 1860 and 292 in 1870.

Bibliography

Szpiro (2010): *Numbers Rule*.

Robinson & Ullman (2010): *A mathematical look at politics*.