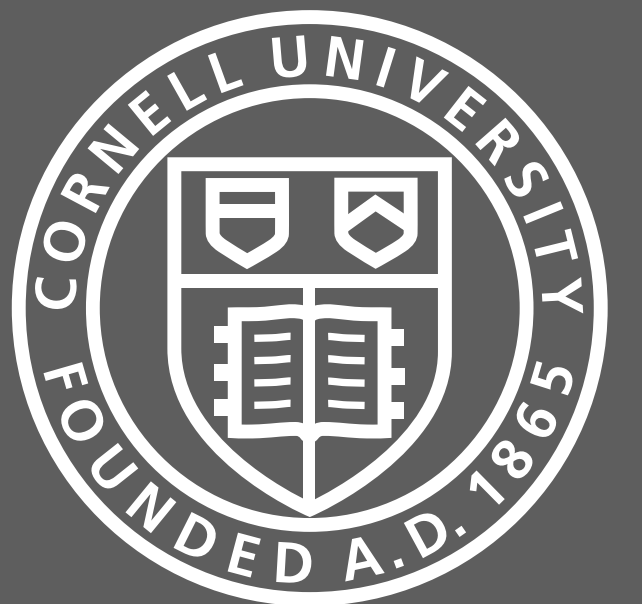


# Utilities & Welfare

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Lecture 15, ORIE 4360/5360, Fall '26



# How We Captured Preferences So Far

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So far, have seen two ways of capturing preferences over alternatives:

- **rankings**:  $a \succ b \succ c$ : voter prefers  $a$  over  $b$  over  $c$ ; we don't know by how much.
- **approval ballots**:  $\{a, b\}$ : Essentially, a ranking with indifferences  $a = b \succ c = \dots$

These preferences are called **ordinal** because they tell us about voter's order of preferences, not about intensity of preference.

# Cardinal Preferences

Another model for preferences: **cardinal** utilities.

Each voter  $i$  has a **utility**  $u_i(x) \in \mathbb{R}$  for each alternative  $x$ , which measures the amount of happiness/satisfaction they get if  $x$  wins.



utility -2



utility 0



utility 2



utility 5

# Advantages of Cardinal Utilities

Tell us about intensity of preferences.

Allow us to compare utilities across voters and alternatives:

Alternative  $a$ :



utility 5

utility 0

Alternative  $b$ :



utility 0

utility 4

# Welfarism

Say we have voters' utilities for all alternatives.

**Welfarism** is a decision-making framework saying that, in choosing between alternatives, we should just consider the vectors of utilities arising from the different alternatives.

So, then, how should we choose between these vectors of utilities? Which objective to optimize?

# Question

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|     | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|-----|-------|-------|-------|-------|
| $a$ | 3     | 3     | 1     | 4     |
| $b$ | 2     | 3     | 2     | 2     |
| $c$ | 10    | 1     | 0     | 2     |
| $d$ | 3     | 2     | 1     | 3     |



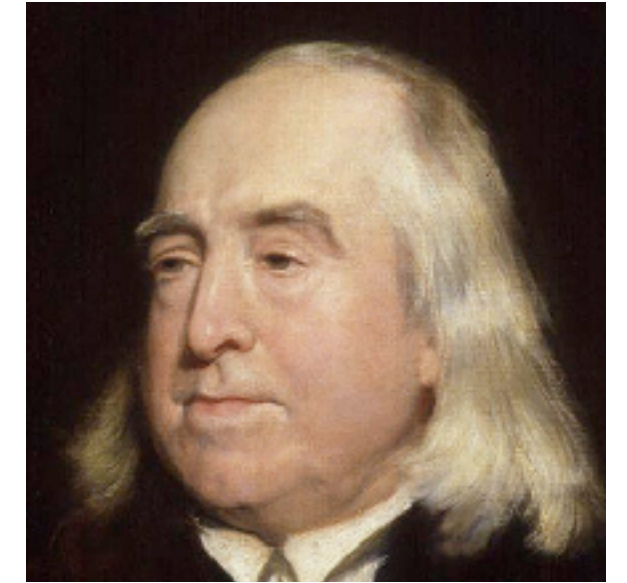
Which alternative would *you* choose?

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# Approach 1: Utilitarianism

The **utilitarian welfare** of alternative  $x$  is the *sum* of all voters' utilities:  $\sum_{i \in N} u_i(x)$ .

Utilitarianism says to pick alternative with highest utilitarian welfare.



Jeremy Bentham  
(1747–1832)

|     | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
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| $c$ | 10    | 1     | 0     | 2     |
| $d$ | 3     | 2     | 1     | 3     |

# Advantages and Disadvantages

## Advantages:

- Appealing to maximize total happiness: "the greatest happiness of the greatest number" (Bentham).

## Disadvantages:

- Tendency to "sacrifice" some voters for the benefit of others.

# Approach 2: Egalitarianism

Compare alternatives through the **leximin social welfare order**:

- Consider the utility vectors for  $x$  and  $y$ :  
 $(u_1(x), \dots, u_n(x))$  and  $(u_1(y), \dots, u_n(y))$ .
- Sort both vectors in increasing order:  
 $u_{(1)}(x) \leq u_{(2)}(x) \leq \dots \leq u_{(n)}(x)$  and  
 $u_{(1)}(y) \leq u_{(2)}(y) \leq \dots \leq u_{(n)}(y)$ .

|     |  |  |  |  |  |
|-----|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------|
| $x$ | 4                                                                                   | 3                                                                                   | 7                                                                                   | 2                                                                                   | 3                                                                                   |
| $y$ | 4                                                                                   | 2                                                                                   | 4                                                                                   | 3                                                                                   | 5                                                                                   |

# Approach 2: Egalitarianism

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Compare alternatives through the **leximin social welfare order**:

- Consider the utility vectors for  $x$  and  $y$ :  $(u_1(x), \dots, u_n(x))$  and  $(u_1(y), \dots, u_n(y))$ .
- Sort both vectors in increasing order:  
 $u_{(1)}(x) \leq u_{(2)}(x) \leq \dots \leq u_{(n)}(x)$  and  
 $u_{(1)}(y) \leq u_{(2)}(y) \leq \dots \leq u_{(n)}(y)$ .
- If  $u_{(1)}(x) > u_{(1)}(y)$ ,  $x$  is better. If  $u_{(1)}(x) < u_{(1)}(y)$ ,  $y$  is better. If they're equal, compare  $u_{(2)}(x)$  with  $u_{(2)}(y)$ . If those are equal  $u_{(3)}(x)$  with  $u_{(3)}(y)$ , etc.

|     |   |   |   |   |   |
|-----|---|---|---|---|---|
| $x$ | 2 | 3 | 3 | 4 | 7 |
|     |   |   | ^ |   |   |
| $y$ | 2 | 3 | 4 | 4 | 5 |

**Egalitarianism**  
chooses best  
alternative in  
this order.

# Question

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|     | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|-----|-------|-------|-------|-------|
| $a$ | 3     | 3     | 1     | 4     |
| $b$ | 2     | 3     | 2     | 2     |
| $c$ | 10    | 1     | 0     | 2     |
| $d$ | 3     | 2     | 1     | 3     |



Which alternative is best according to egalitarianism?

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# Advantages and Disadvantages

## Advantages:

- Focus on compensation/fairness/those least well off.

## Disadvantages:

- A tiny improvement in utility for the worst off is worth sacrificing unlimited amounts of utility for others.

# Approach 3: Nash Social Welfare

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A middle ground (applicable if utilities are nonnegative): choose the alternative  $x$  that maximizes the **Nash social welfare**, i.e., the product of utilities  $\prod_{i \in N} u_i(x)$ .

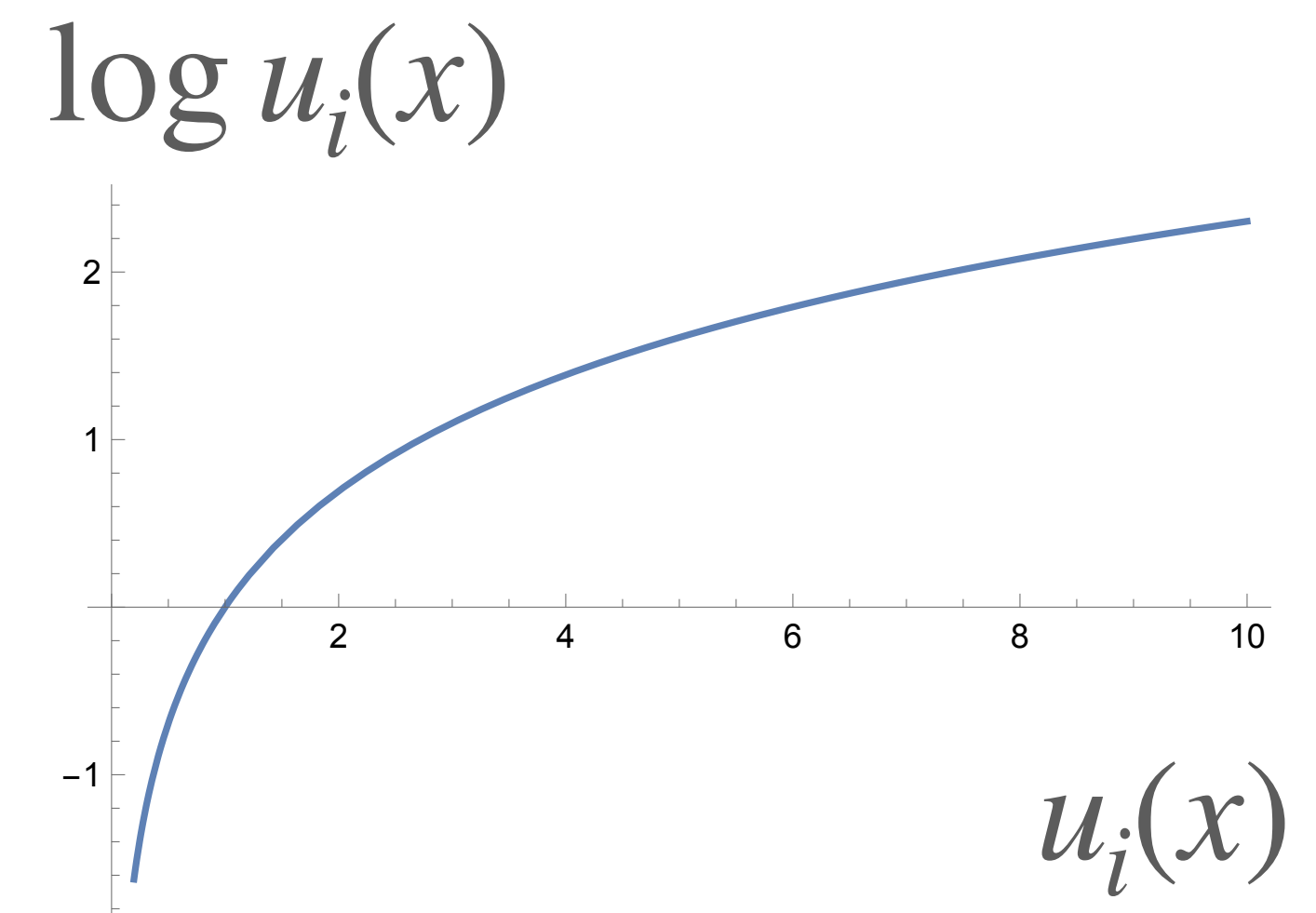
|     | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
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# Log-Formulation

Instead of maximizing the product of utilities, often useful to maximize the logarithm:

$$\log \prod_{i \in N} u_i(x) = \sum_{i \in N} \log u_i(x).$$

This helps us understand Nash welfare: any voter's increase in utility increase the Nash welfare, but larger effect if voter has low utility than if they already have high utility.



# Advantages and Disadvantages

## Advantages:

- Still extremely sensitive to utilities close to zero, encourages no utility to be too low.
- Also still sensitive to higher utilities.

## Disadvantages:

- Strikes one particular compromise between "fitness" and "compensation", no free lunch.

# Would We Ever Select $d$ ?

|     | $u_1$ | $u_2$ | $u_3$ | $u_4$ |
|-----|-------|-------|-------|-------|
| $a$ | 3     | 3     | 1     | 4     |
|     | II    | VI    | II    | VI    |
| $d$ | 3     | 2     | 1     | 3     |

# Pareto Optimality

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An alternative  $x$  **Pareto dominates** alternative  $y$  if all voters weakly prefer  $x$  over  $y$  and at least one voter strictly prefers  $x$  over  $y$ .

An alternative  $x$  is called **Pareto optimal** if it is not Pareto dominated by any other alternative.

# Question



Let  $x$  be an alternative. Which are true?

- If  $x$  is utilitarian-optimal,  $x$  is Pareto optimal.
- If  $x$  is egalitarian-optimal,  $x$  is Pareto optimal.
- If  $x$  is Nash-optimal,  $x$  is Pareto optimal.

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# Nitpicks About Nash Welfare

We actually have to be a bit more careful with Nash welfare when all alternatives have someone with utility zero:

- Prefer alternatives that give fewer people zero utility.
- Among those with lowest number of zeros, pick the alternative for whom the product of nonzeros is largest.



# Cardinal or Ordinal Preferences?

In some settings, one can define reasonable “objective” notions of cardinal preferences: e.g., probability that medical operation succeeds, money received, willingness to pay, ....

In general, though, it's questionable to put different people's values/preferences on a common scale. Because of this, “interpersonal comparisons of utility” are controversial.

# Cardinal or Ordinal Preferences?

Some people advocate for eliciting cardinal information in voting ("range voting"):

| Governor Candidates |   | Score <i>each</i> candidate by filling a number (0 is worst; 9 is best) |
|---------------------|---|-------------------------------------------------------------------------|
| 1: Candidate A      | → | <b>0</b> ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ ⑨                                              |
| 2: Candidate B      | → | ① ② ③ ④ ⑤ ⑥ ⑦ ⑧ <b>9</b>                                                |
| 3: Candidate C      | → | ① ② ③ ④ ⑤ ⑥ <b>7</b> ⑧ ⑨                                                |

Issues: hard to elicit, big incentives for manipulation.

# A Tool for Analyzing Ordinal Voting?

Cardinal utilities are much more expressive than rankings/approvals, for example because they can express intensity.

But we also discussed reasons not to directly ask voters for cardinal utilities: difficulty of elicitation and big opportunities for manipulation.

**Next up:** Analyze ranked-choice elections through the possible cardinal utilities behind the rankings.

Moulin (2004): *Fair Division and Collective Welfare*. Chapters 1.2 and 3.