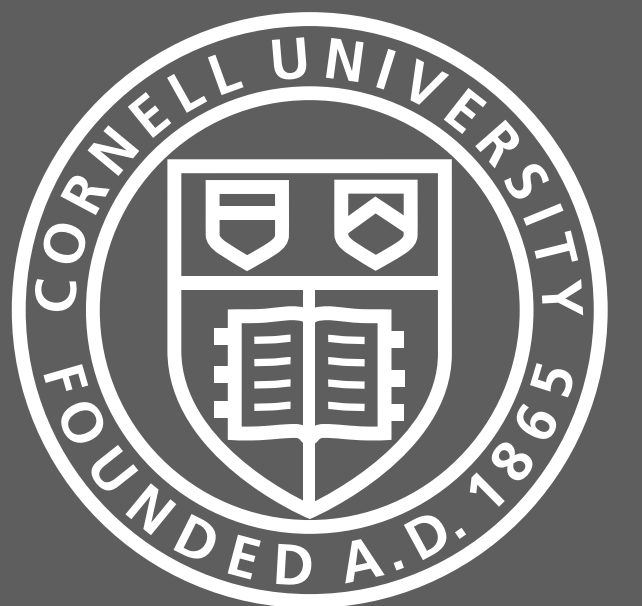


Proportionality in Multi-Winner Elections

Paul Gözl

Lecture 12, ORIE 4360/5360, Fall '26



Recap: Voting Model

2

Voters $N = \{1, 2, \dots, n\}$. **Alternatives** M , $|M| = m$.

Each voter i has an **approval set** $A_i \in \mathcal{P}^*(M)$ of alternatives;
preference profile $\mathbf{A} \in \mathcal{P}^*(M)^n$ is a collection of all voters' approval sets.

Committee size $k \in \mathbb{N}$. Write
 $\binom{M}{k} := \left\{ W \in \mathcal{P}(M) \mid |W| = k \right\}$ for the set of **committees**.

A **social choice function** is a function $f: \mathcal{P}^*(M)^n \rightarrow \binom{M}{k}$.

For profile $\mathbf{A}$, committee $W \in \binom{M}{k}$, set $u_i(W) := |W \cap A_i|$.

Recap: Thiele Methods

3

For $w_1, w_2, \dots, w_k \in \mathbb{R}_{\geq 0}$, select committee
$$\operatorname{argmax}_{W \in \binom{M}{k}} \sum_{i \in N} w_1 + w_2 + \dots + w_{u_i(W)}.$$



Thorvald Thiele
(1838–1910)

Examples:

- Approval Voting: $w_t = 1$.
- Chamberlin–Courant: $w_1 = 1, w_t = 0$ ($t > 1$).
- Proportional Approval Voting (PAV): $w_t = 1/t$:
maximize $\sum_{i \in N} H_{u_i(W)}$, where H_t is the t th harmonic number.

Recap: PAV in Party-List Settings

Consider a “party-list” setting (i.e., voters are split into blocks, blocks approve disjoint alternatives, each block approves $\geq k$ alternatives).

Then, for any block with $x \geq \ell n/k$ voters ($\ell \in \mathbb{N}$), PAV selects at least ℓ of their alternatives.

What should we require when voters' approvals don't have the block structure?

Axioms for Proportionality?

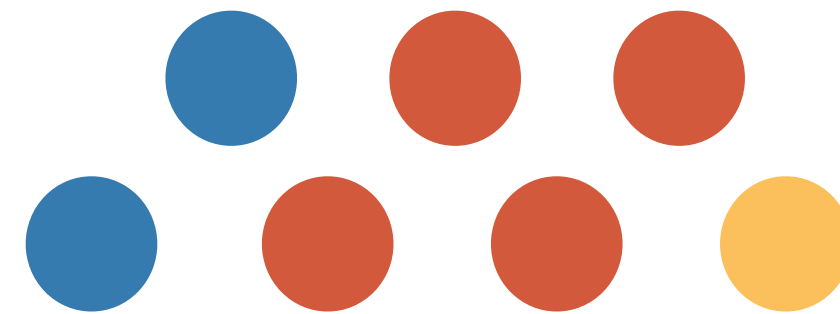
5



Excellence

select k most popular
alternatives

application:
shortlisting



Proportionality

represent opinions *in
proportion* to their support

application: parliamentary
elections, committee or
board elections



Diversity

represent *many*
different opinions

application: product
selection

Let's Start With $\ell = 1$

6

Idea: Each group of size n/k deserves representation by one alternative.

🤔 If a group likes no alternative in common, how would we even represent?

Definition. A group $S \subseteq N$ of voters is called **1-cohesive** if $|\bigcap_{i \in S} A_i| \geq 1$.

$$n = 36, k = 3$$

$$10 \times \{a, c, e\}$$

$$9 \times \{a, b, e\}$$

$$1 \times \{a, b\}$$

$$8 \times \{b, d\}$$

$$8 \times \{c, d\}$$

An Axiom That Is Too Strong

7

Next attempt: For any 1-cohesive group S of size $\geq n/k$, one of their jointly approved alternatives must be elected (i.e., $|W \cap \bigcap_{i \in S} A_i| \geq 1$).

🤔 often not satisfiable:

$$\begin{array}{l} n = 4, k = 2 \quad \{a, b\} \\ \quad \quad \quad \quad \quad \{b, c\} \\ \quad \quad \quad \quad \quad \{c, d\} \\ \quad \quad \quad \quad \quad \{a, d\} \end{array}$$

Justified Representation (JR)

Definition. A committee W satisfies **justified representation (JR)** if, for any 1-cohesive group S of size $\geq n/k$, $u_i(W) \geq 1$ for some $i \in S$.

A social choice function satisfies **JR** if it returns, for any input profile, a JR committee.

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Which out of AV, CC, and PAV satisfy JR?

pollev.com/goelz

Recap: Tyranny of the Minority

10

$n = 100, k = 10$

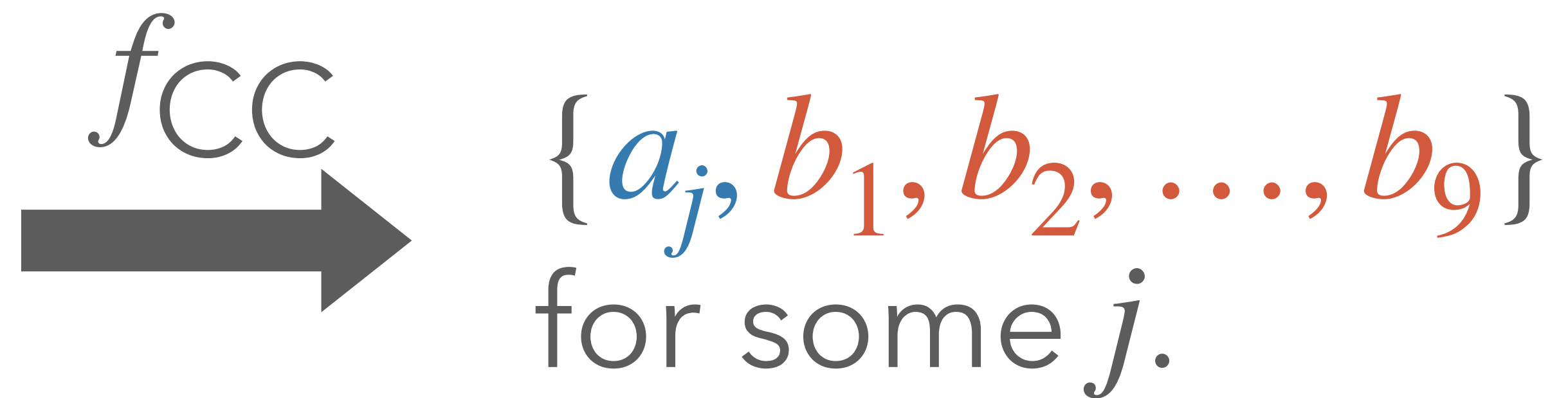
91x $\{a_1, a_2, \dots, a_{10}\}$

1x $\{b_1\}$

1x $\{b_2\}$

...

1x $\{b_9\}$



On this profile, JR has little bite: Only says we need to select some a_j . Some groups deserve representation by several alternatives!

Extended Justified Repr. (EJR)

11

Definition. For $1 \leq \ell \leq k$, a group $S \subseteq N$ is called ℓ -cohesive if $|\bigcap_{i \in S} A_i| \geq \ell$.

Definition. A committee W satisfies **extended justified representation (EJR)** if, for all ℓ -cohesive groups S of size $\geq \ell \cdot n/k$, $u_i(W) \geq \ell$ for some $i \in S$.

A social choice function satisfies **EJR** if it returns, for any input profile, an EJR committee.

Observations

EJR implies JR, since JR is just the special case where $\ell = 1$.

Hence, we immediately know that AV fails EJER.
CC failed EJER on “tyranny of minority” profile.

PAV Satisfies EJR

Theorem [Aziz et al. 2017]. PAV Satisfies EJR.
[Proof on blackboard.]

A Price-Based Voting Rule

14

PAV is great, but hard to compute.

Also hard to explain the outcome: Optimality is not obvious.

Interesting alternative approach, due to Phragmén (1895): voters have money, spend it on alternatives they like.

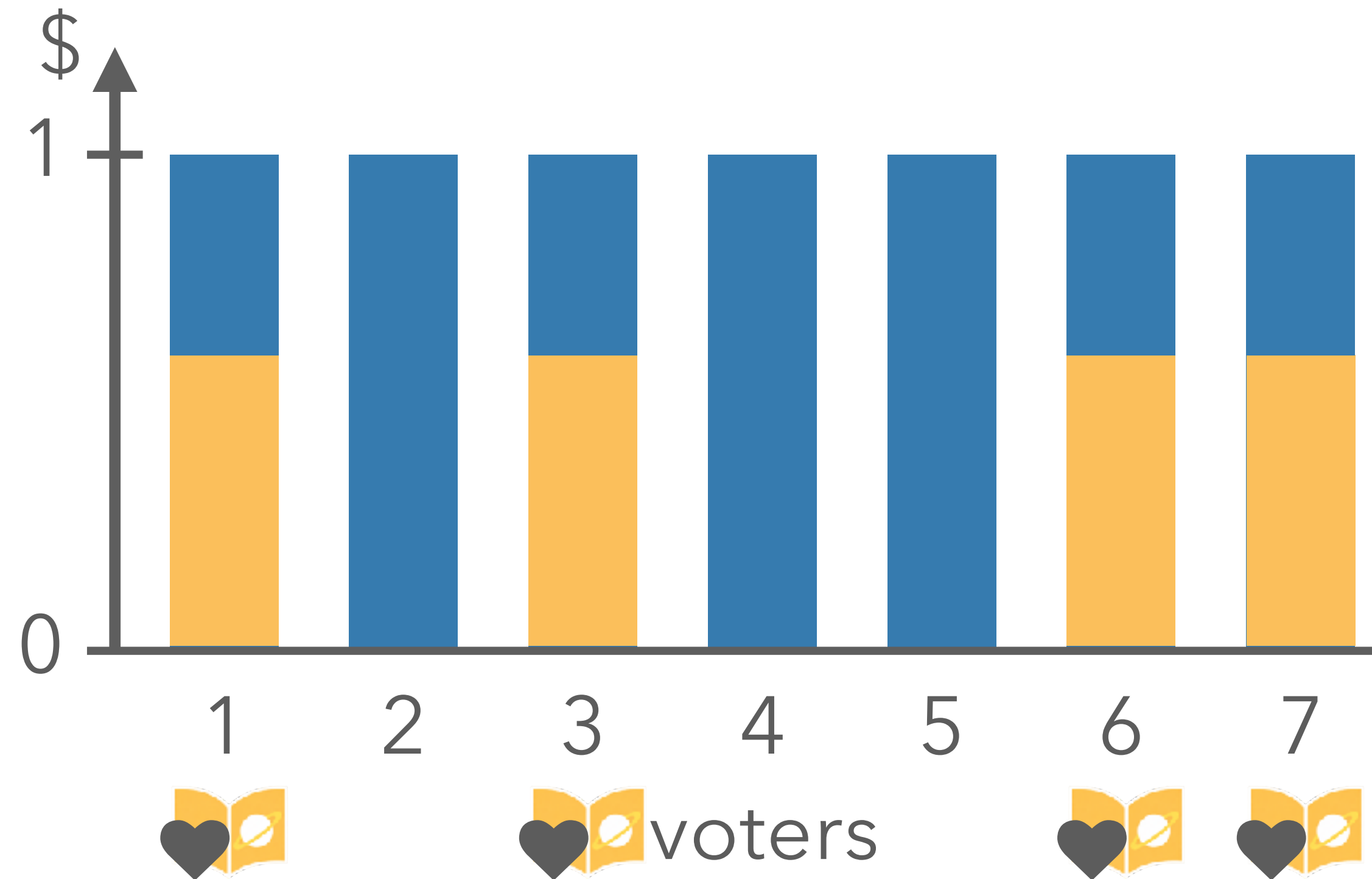
I'll directly show you a voting rule based on this idea that satisfies EJR.



Edvard Phragmén
(1863–1937)

The Method of Equal Shares (MES)

15



Each voter starts with \$1, adding an alternative costs $\$n/k$.

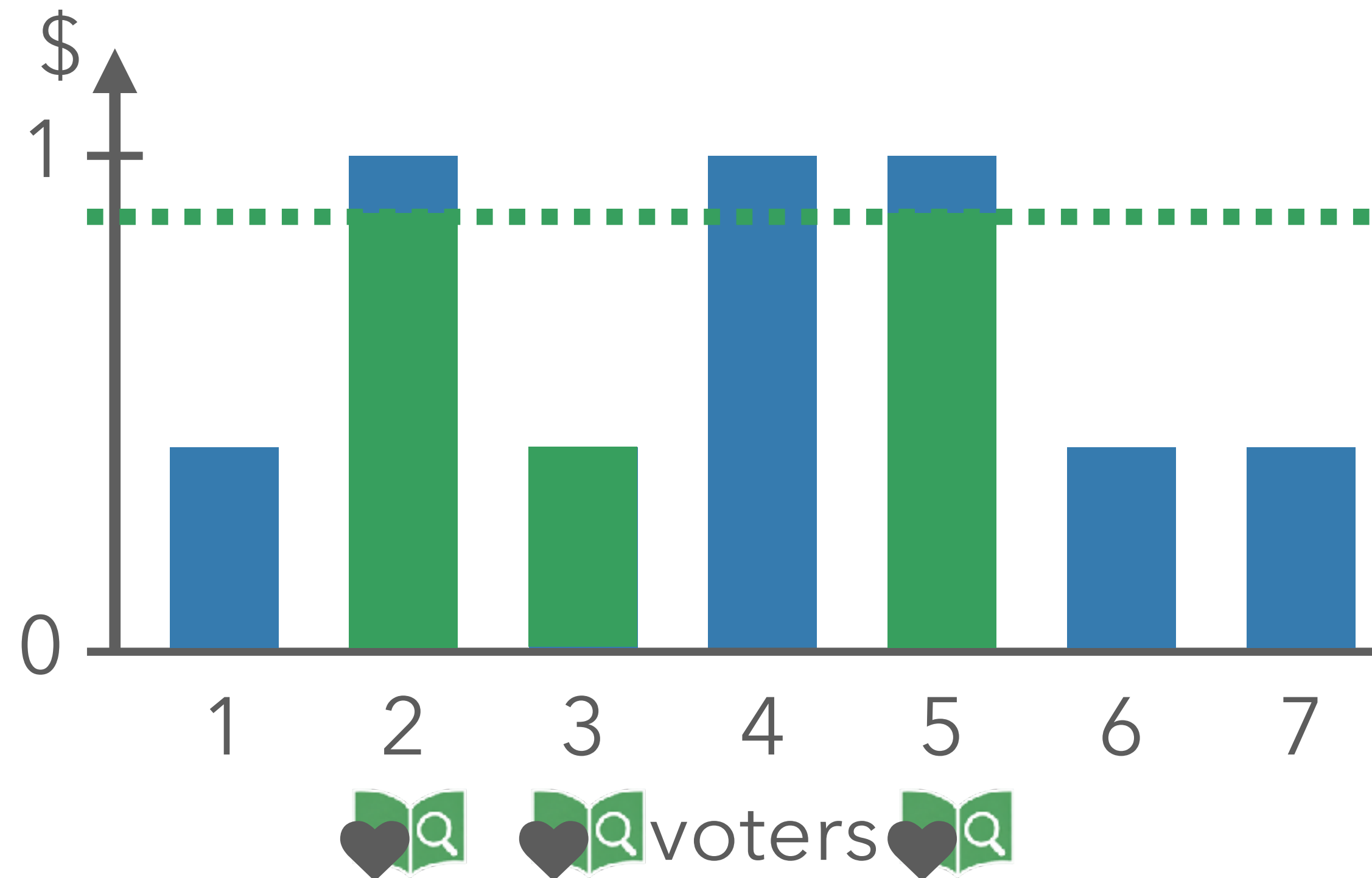
Grow W one by one. Voters only pay for alternatives they like.

$$n = 7, k = 3$$

$$W = \{ \quad \}$$

The Method of Equal Shares (MES)

16



$$n = 7, k = 3$$

$$W = \{ \text{📖} \}$$

Each voter starts with \$1, adding an alternative costs $\$n/k$.

Grow W one by one. Voters only pay for alternatives they like.

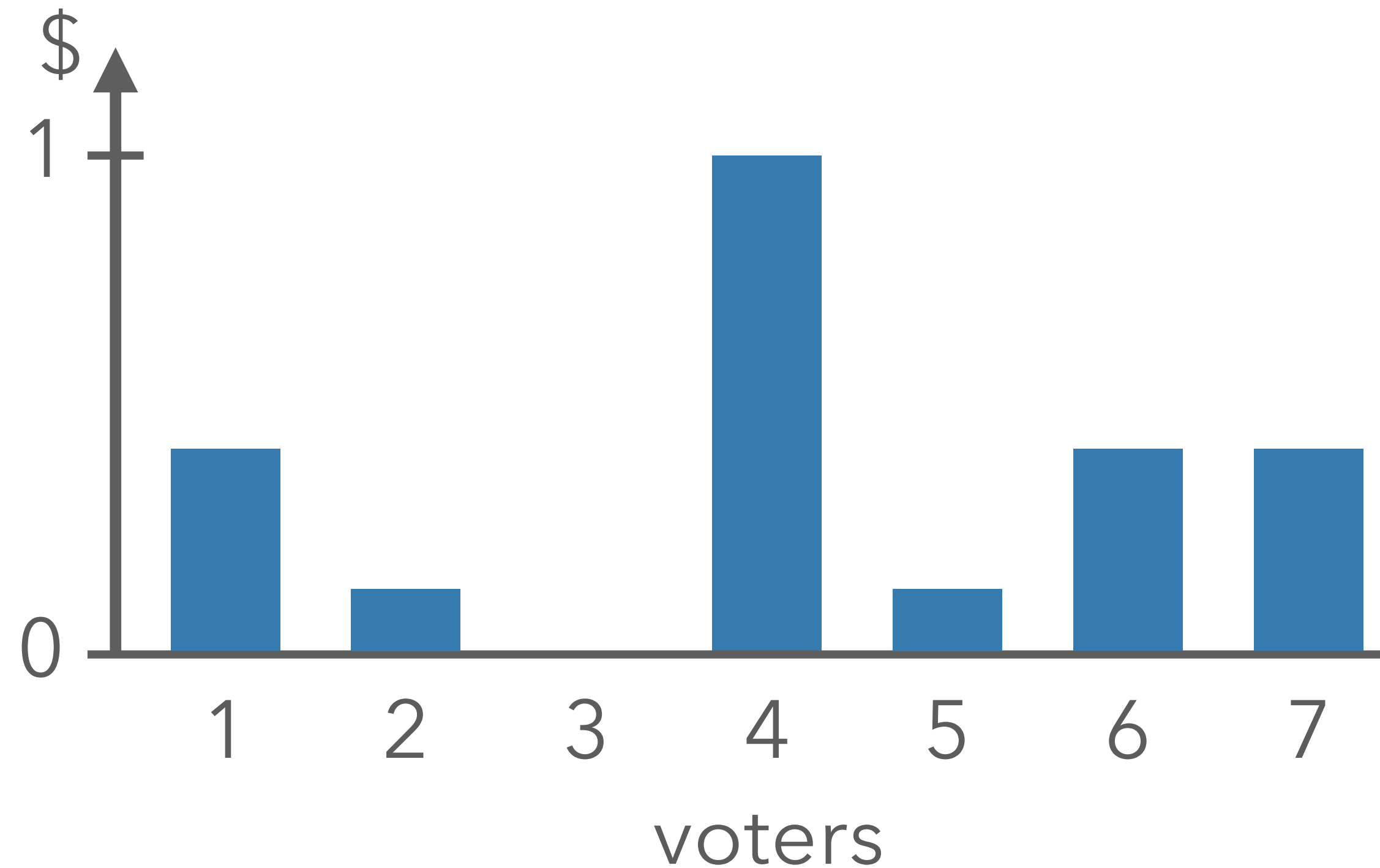
Keep going until no alternative can be afforded any more.

Which alternative to add next and who pays how much?

MES: each voter pays the same amount (or all their remaining money, whichever is less). Add alternative with lowest maximum payment.

The Method of Equal Shares (MES)

17



$$n = 7, k = 3$$

$$W = \{ \text{📖} \text{📖} \}$$

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The Method of Equal Shares (MES)

MES can never add more than k alternatives.

It can add fewer than k , though, in which case we fill the rest of the committee with some other process.

Theorem [Peters & Skowron 2020]. The method of equal shares satisfies EJR, no matter how it is completed.

Let's Calculate MES Together

$$n = 6, k = 3$$

$$A_1 = \{a, b\}$$

$$A_2 = \{a, c, d\}$$

$$A_3 = \{a, c\}$$

$$A_4 = \{a, d\}$$

$$A_5 = \{b, c\}$$

$$A_6 = \{b\}$$

Is EJR Enough?

20

$n = 20, k = 10$

3x { a, c }
3x { b, c }

f_{PAV}
→ { $c, d_1, d_2, \dots, d_9$ }

14x { $d_1, d_2, \dots, d_9$ }

This group is of size $3 \cdot n/k$, but only 1-cohesive.
Would have preferred { a, b, c } over outcome.

Idea: No group should be able to come up with a “better” subcommittee.

Definition. A committee W satisfies **core stability** if, for all groups S of size $\geq \ell \cdot n/k$, there is no $T \subseteq A$, $|T| = \ell$, such that $u_i(T) > u_i(W)$ for all $i \in S$.

A social choice function satisfies **core stability** if it always returns core stable committees.

Observation: Generalizes EJR.

Definition. A committee W satisfies **core stability** if, for all groups S of size $\geq \ell \cdot n/k$, there is no $T \subseteq A$, $|T| = \ell$, such that $u_i(T) > u_i(W)$ for all $i \in S$.

A social choice function satisfies **core stability** if it always returns core stable committees.

Big open question in last 10 years: Does a core stable committee always exist? Answer next time.

Aziz et al.: *Justified Representation in Approval-Based Committee Voting*. Social Choice and Welfare (2017).

Peters & Skowron: *Proportionality and the Limits of Welfarism*. ACM EC (2020).