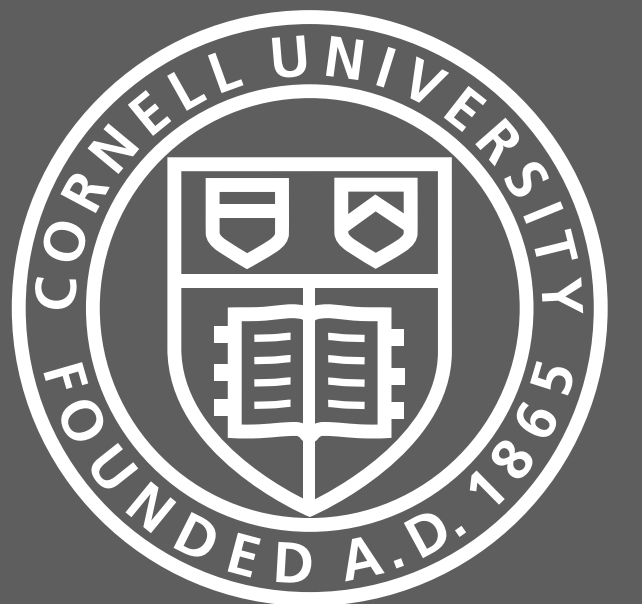


Multi-Winner Elections

Paul Gözl

Lecture 11, ORIE 4360/5360, Fall '26



Committee Elections

2

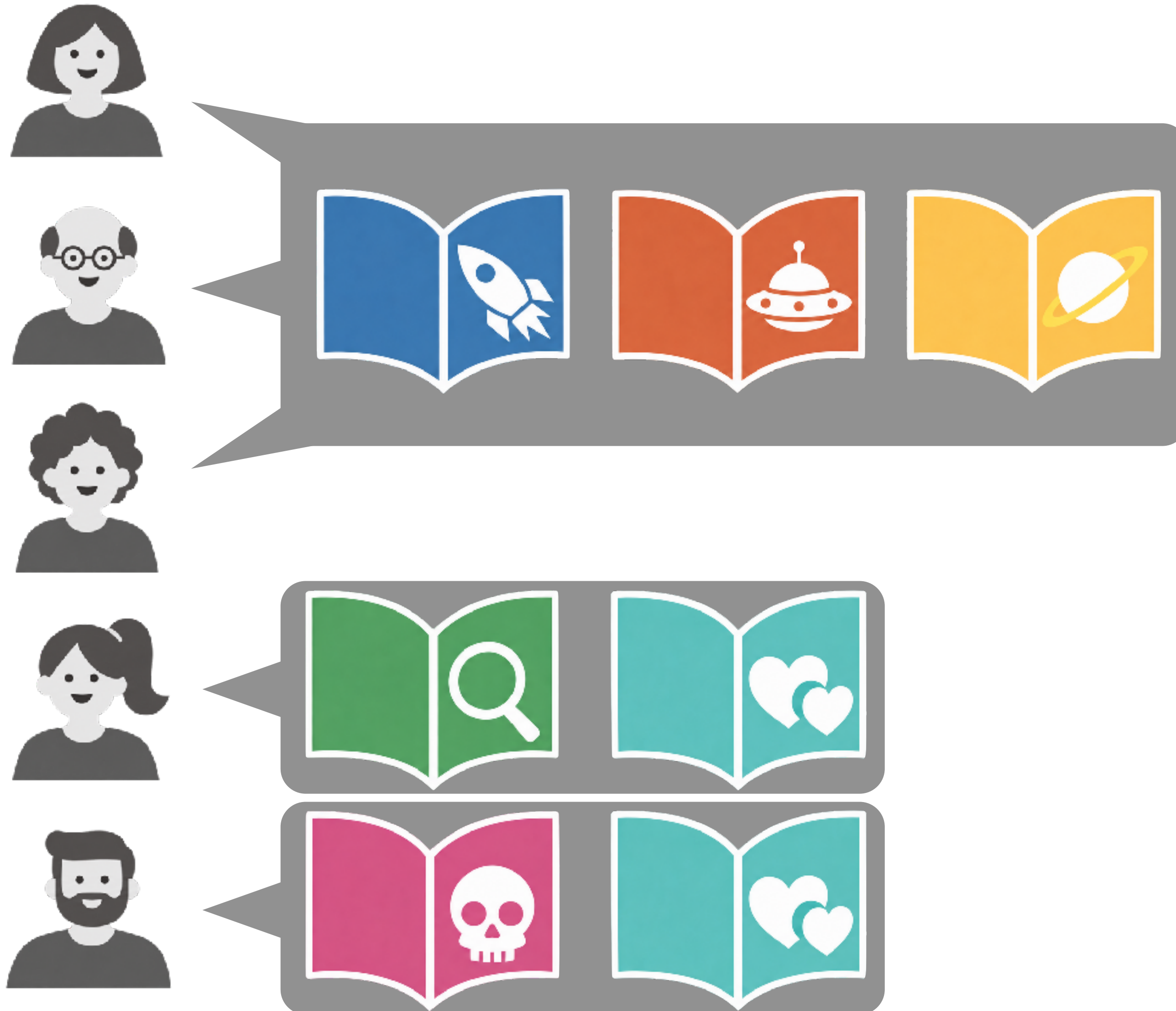
So far, the outcome of all our elections was a single alternative. Sometimes, we want to choose several alternatives:

- city council
- bands for Slope day
- books to read in book club



How To Select Three Books To Read?

3



Approval-Based Multi-Winner Model

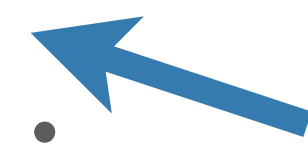
4

Voters $N = \{1, 2, \dots, n\}$. **Alternatives** M , $|M| = m$.

Each voter i has an **approval set** $A_i \in \mathcal{P}^*(M)$ of alternatives; **preference profile** $\mathbf{A} \in \mathcal{P}^*(M)^n$ is a collection of all voters' approval sets.

Committee size $k \in \mathbb{N}$. Write $\binom{M}{k} := \left\{ W \in \mathcal{P}(M) \mid |W| = k \right\}$ for the set of **committees**.

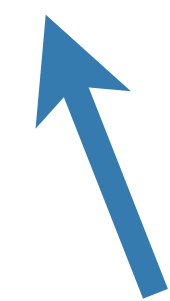
A **social choice function** is a function

$f : \mathcal{P}^*(M)^n \rightarrow \binom{M}{k}$.  Back to *resolute* rules. Voting rule returns a set of candidates, but these k candidates are definitely selected, not just tied for winning.

Multi-Winner Approval Voting (AV)

For committee $W \in \binom{M}{k}$, write $u_i(W) := |W \cap A_i|$.

$$f_{AV}(A) := \operatorname{argmax}_{W \in \binom{M}{k}} \sum_{i \in N} u_i(W).$$



Because resolute rule, this argmax returns a single set W , breaking ties in some way.

This is the same as choosing the k alternatives with highest approval score.

Tyranny of the Majority



Tyranny of the Majority

$$n = 100, k = 10$$

$$\begin{array}{l} 51 \times \{a_1, a_2, \dots, a_{10}\} \\ 49 \times \{b_1, b_2, \dots, b_{10}\} \end{array} \xrightarrow{f_{AV}} \{a_1, a_2, \dots, a_{10}\}$$

Chamberlin–Courant (CC)

Alternative idea: Maximize the number of voters who like *something* in the outcome.

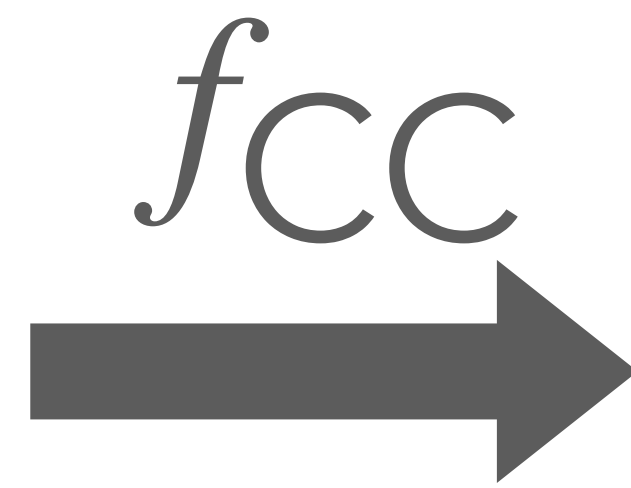
$$f_{CC}(A) := \operatorname{argmax}_{W \in \binom{M}{k}} \left| \{i \in N \mid u_i(W) \geq 1\} \right|.$$

Chamberlin-Courant on Examples

$n = 100, k = 10$

51x $\{a_1, a_2, \dots, a_{10}\}$

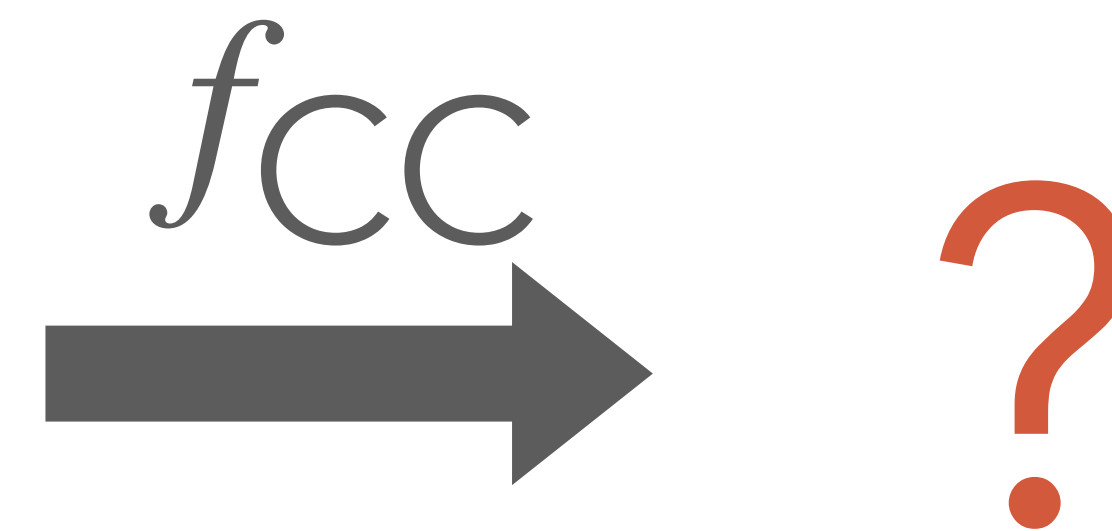
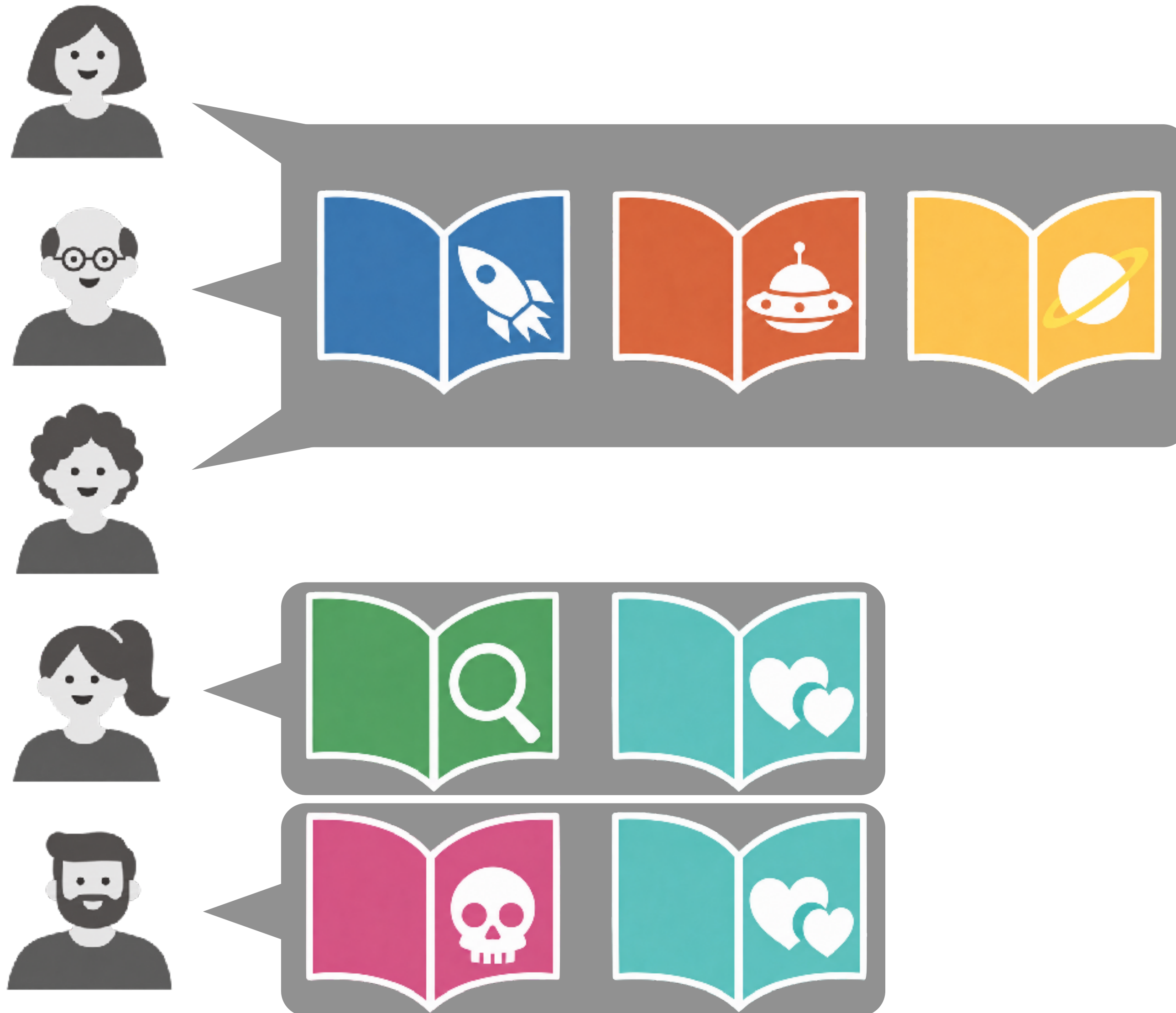
49x $\{b_1, b_2, \dots, b_{10}\}$



All committees that
include some a_j
and some $b_{j'}$ are
tied...

Chamberlin-Courant on Examples

10



Tyranny of the Minority

$$n = 100, k = 10$$

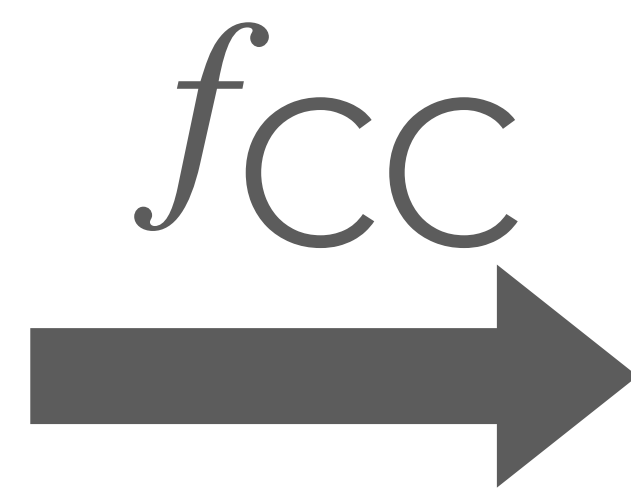
$$91 \times \{a_1, a_2, \dots, a_{10}\}$$

$$1 \times \{b_1\}$$

$$1 \times \{b_2\}$$

...

$$1 \times \{b_9\}$$



$$\{a_j, b_1, b_2, \dots, b_9\}$$

for some j .

Three Kinds of Representation

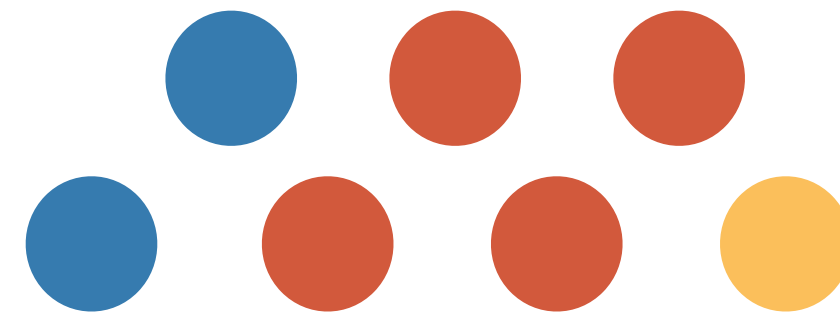
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Excellence

select *k* most popular
alternatives

application:
shortlisting



Proportionality

represent opinions *in
proportion* to their support

application: parliamentary
elections, committee or
board elections



Diversity

represent *many*
different opinions

application: product
selection

Question

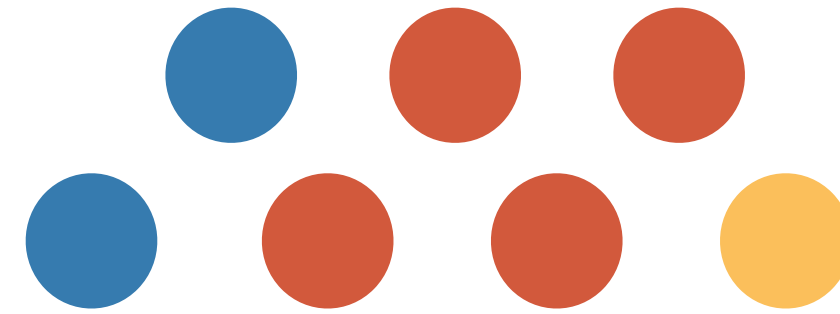
13



Excellence

select k most popular alternatives

application: shortlisting



Proportionality

represent opinions *in proportion* to their support

application: parliamentary elections, committee or board elections



Diversity

represent *many* different opinions

application: product selection

Q: Which type of representation does multi-winner approval voting (AV) pursue?

Question

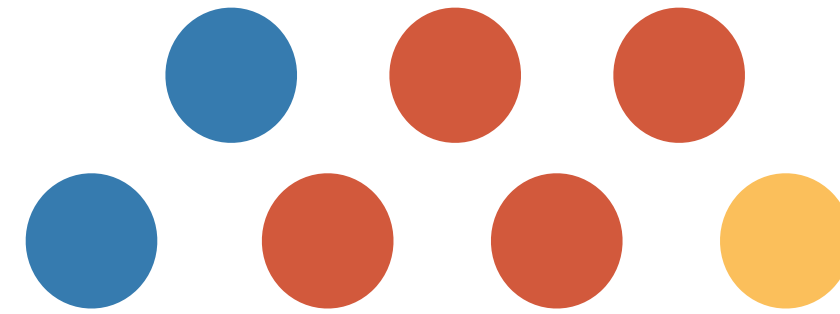
14



Excellence

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application:
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Proportionality

represent opinions *in proportion* to their support

application: parliamentary elections, committee or board elections



Diversity

represent *many* different opinions

application: product selection

Q: Which type of representation does Chamberlin-Courant (CC) pursue?

Question

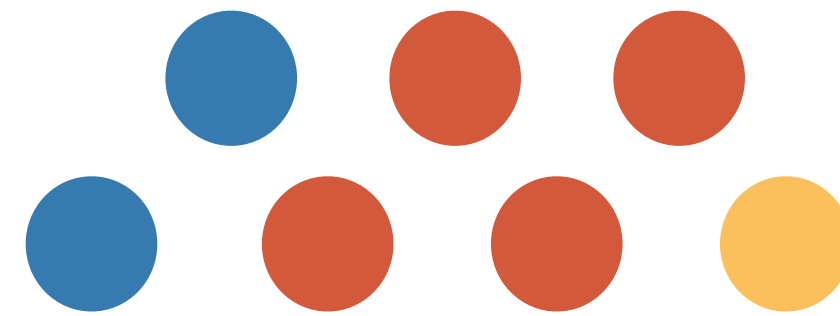
15



Excellence

select k most popular alternatives

application:
shortlisting



Proportionality

represent opinions *in proportion* to their support

application: parliamentary elections, committee or board elections



Diversity

represent *many* different opinions

application: product selection

Q: How would you build a voting rule for proportionality?

Thiele Methods

16

For $w_1, w_2, \dots, w_k \in \mathbb{R}_{\geq 0}$, select committee
$$\operatorname{argmax}_{W \in \binom{M}{k}} \sum_{i \in N} w_1 + w_2 + \dots + w_{u_i(W)}.$$



Thorvald Thiele
(1838–1910)

Examples:

- Approval Voting: $w_t = 1$.
- Chamberlin–Courant: $w_1 = 1$, $w_t = 0$ ($t > 1$).
- **Proportional Approval Voting (PAV)**: $w_t = 1/t$:
maximize $\sum_{i \in N} H_{u_i(W)}$, where H_t is the t th harmonic number.

Why Harmonic Numbers in PAV?

$$n = 22, k = 11$$

6x { $a_1, a_2, \dots$ }	6	3	2	1.5	1.2	1
4x { $b_1, b_2, \dots$ }	4	2	1.33	1	0.8	0.67
10x { $c_1, c_2, \dots$ }	10	5	3.33	2.5	2	1.67
2x { $d_1, d_2, \dots$ }	2	1	0.66	0.5	0.4	0.33

$$W = \{\}$$

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$$W = \{c_1, \}$$

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$$W = \{ c_1, a_1, c_2, b_1, c_3, \}$$

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$$W = \{c_1, a_1, c_2, b_1, c_3, a_2, c_4, a_3, b_2, c_5, d_1\}$$

Why Did This Work Out?

In such a “party-list” setting (i.e., voters are split into blocks, blocks approve disjoint alternatives, each block approves $\geq k$ alternatives), suppose some block has $x \geq \ell \cdot n/k$ voters for $\ell \in \mathbb{N}$.

Then, PAV selects $\geq \ell$ of this block’s alternatives:

block’s entries in table are $\frac{x}{1} \geq \frac{x}{2} \geq \dots \geq \frac{x}{\ell} \geq \frac{n}{k}$.

So, conclusion follows if we select all table entries that are at least n/k .

Why Harmonic Numbers in PAV?

$$n = 22, k = 11$$

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$$W = \{c_1, a_1, c_2, b_1, c_3, a_2, c_4, a_3, b_2, c_5, d_1\}$$

Why Did This Work Out (Cont.)

27

Suppose that for each block j with n_j voters, we selected t_j entries and all had value $\geq n/k$.

Hence, $n_j/t_j \geq n/k$ and $t_j \leq n_j \cdot k/n$.

Suppose for contradiction that these inequalities are strict for some block.

Then, $\sum_{\text{blocks } j} t_j < \sum_j n_j \cdot k/n = k/n \cdot n = k$.

Contradiction, so must have selected all entries above n/k , and $\geq \ell$ alternatives for our block.

Why Harmonic Numbers?

In fact, harmonic weights are the only weights (up to scaling) such that Thiele's method is proportional in party-list setting.

Later in this class, we will revisit the party-list setting, but next we will focus on proportionality if approval sets intersect.

Kilgour: *Approval Balloting for Multi-Winner Elections* (Chapter 6 in Laslier & Sanver: *Handbook on Approval Voting*, 2010).