Epistemic Models for Voting

+ Review

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Review 1: Head-To-Head Elections

1	2	3	4	5
c	b	b	a	c
a	c	c	c	a
b	a	a	b	b

In the profile above, does a or b win in a head-to-head election?

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Review 2: Copeland

1	2	3	4	5
b	a	b	e	d
a	b	d	d	a
c	c	e	a	b
d	e	c	b	e
e	d	a	c	c

Which alternatives have the highest Copeland score?

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Review 3: Neutrality & Anonymity

1	2	3	4	5
a	a	b	c	d
d	d	d	d	c
b	c	c	b	b
c	b	a	a	a

$\xrightarrow{f}$ b

1	2	3	4	5
c	b	a	d	c
b	d	b	b	b
d	a	d	a	a
a	c	c	c	d

If f is anonymous and neutral, which alternative must win on the right?

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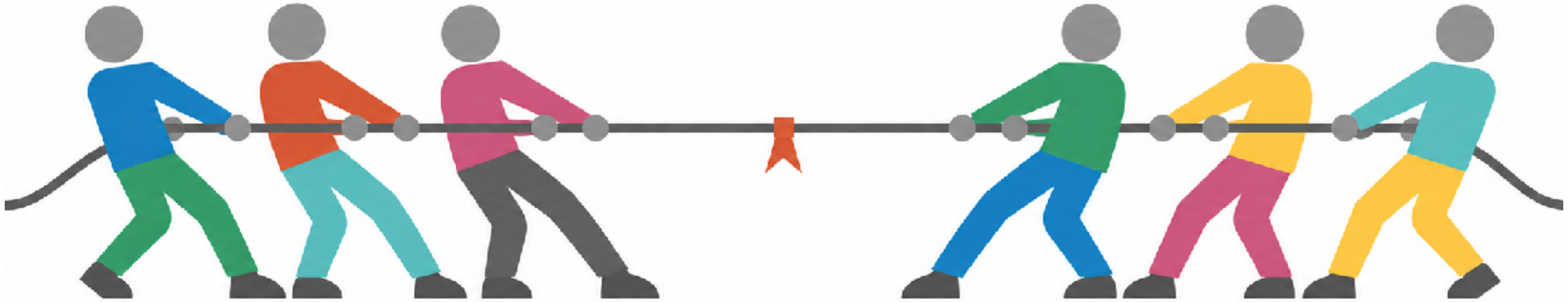
Two Conceptions of Voting

Often: voters have different interests and preferences, voting should find compromise.



Two Conceptions of Voting

Often: voters have different interests and preferences, voting should find compromise.

Other conception: there is a “best” ranking or alternative. All voters want to identify it, but might be mistaken in what it is. Voting should extract signal from the noise.

Q: In which settings can we talk about an objectively “best” alternative?

Condorcet's Jury Theorem

Actually, much of Condorcet's famous text about voting was about such epistemic settings.

Nicolas de Condorcet (1743–1794)

For now, consider $m = 2$ case:

W.l.o.g., a is the good alternative, b the bad one.

Each of n voters independently votes for a with probability $1/2 < p \leq 1$ or for b with probability $1 - p$.

Condorcet's Jury Theorem

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Theorem [Condorcet 1785]. As $n \to \infty$, the probability that the majority is correct goes to 1.

Proof of Condorcet's Jury Theorem

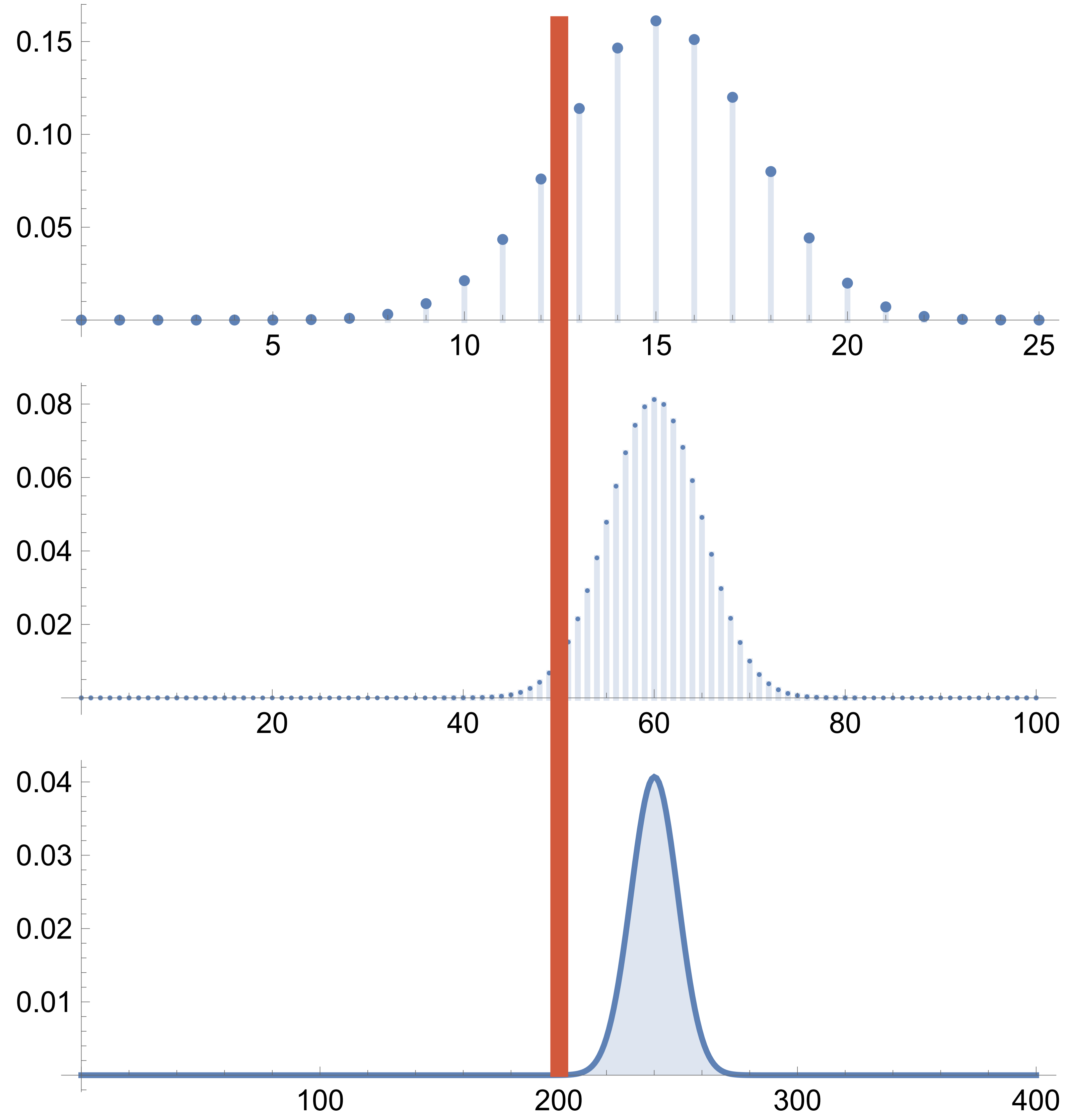
Theorem [Condorcet 1785]. As $n \to \infty$, the probability that the majority is correct goes to 1.

Idea: Law of large numbers/ concentration of measure.

p=60%, n = 25

p=60%, n = 100

p=60%, n = 400



Proof of Condorcet's Jury Theorem

Theorem [Condorcet 1785]. As $n \to \infty$, the probability that the majority is correct goes to 1.

Theorem [Chernoff 1952]. Let $X_1, \ldots, X_n$ be independent random variables with values in $[0,1]$. Let X be their sum, $\mu = \mathbb{E}[X]$, and $\delta > 0$. Then, $\mathbb{P}[X \leq (1-\delta)\mu] \leq e^{-\delta^2\mu/2}$.

Condorcet's Idea for $m \geq 3$

1	2	3	4	5
b	*a*	*b*	*e*	*d*
a	*b*	*d*	*d*	*a*
c	*c*	*e*	*a*	*b*
d	*e*	*c*	*b*	*e*
e	*d*	*a*	*c*	*c*

When majority preferences don't give a ranking/have cycles, that's because people made mistakes in observing the objectively true ranking.

Let's select the ranking that is "most likely" to be this true ranking.

Maximum Likelihood Estimation

Latent parameter: true ranking π.

Generative model

Defines, for each observation y, likelihood $\mathbb{P}[y \mid \pi]$.

Observation y from voter, e.g., their reported ranking σ_i.

We see independent observations $y_1, \ldots y_n$.

For each ground-truth π, we can calculate the likelihood $\prod_{i \in N} \mathbb{P}[y = y_i \mid \pi]$.

Our best guess for what the ground truth is is the maximum likelihood estimator (MLE), i.e., π with largest likelihood.

(Often convenient to compare log of likelihood.)

Condorcet's Model

(1) In every pairwise comparison, each voter chooses the better alternative with some fixed probability p, where $1/2 < p \leq 1$.

(2) Each voter's judgment on every pair of alternatives is independent of her judgment on every other pair.

(3) Each voter's judgment is independent of the other voters' judgments.

(4) Each voter's judgment produces a ranking of the alternatives.

Q: Can you spot the contradiction?

Condorcet’s Model

“The obscurity and self-contradiction are without any parallel, so far as our experience of mathematical works extends … no amount of examples can convey an adequate impression of the evils.”

Isaac Todhunter
(1820–1884)

Fix 1 for Condorcet's Model

(4) ~~Each voter's judgment produces a ranking of the alternatives.~~

Each voter independently guesses each pairwise comparison, and we collect these pairwise preferences $\succ_i$ even if the voter is inconsistent.

(So, just for today, $\succ_i$ can be nontransitive, e.g., $a \succ_i b \succ_i c \succ_i a$. Just denote which alternative is preferred in each pair.)

MLE for Fix 1

Let π be the true ranking, have for each voter i their preferred choice $\succ_i$ among all pairs of alternatives.

We have a generative model, i.e., can compute $\mathbb{P}[\succ_i | \pi]$ for each $\pi, \succ_i$.

Can compute MLE, i.e., $\mathrm{argmax}_\pi \prod_{i \in N} \mathbb{P}[\succ_i | \pi]$.

Fix 1 for Condorcet's Model

If true ranking is $a \succ_\pi b \succ_\pi c$, probability of observing $\succ_1, \ldots, \succ_n$, with aggregate counts on the right:
$p^8(1-p)^5 \cdot p^6(1-p)^7 \cdot p^{11}(1-p)^2$.

	a	b	c
a	–	8	6
b	5	–	11
c	7	2	–

Means 7 voters prefer c over a.

If true ranking is $a \succ_\pi c \succ_\pi b$, then
$p^8(1-p)^5 \cdot p^6(1-p)^7 \cdot p^2(1-p)^{11}$.

$$\mathbb{P}[\{\sigma_i\}_{i \in N} \mid \pi] = p^{\#\text{agree}} \cdot (1-p)^{\#\text{disagree}} \propto \left(\frac{1-p}{p}\right)^{\#disagree}.$$

Insight: Rankings with fewest disagreements has largest likelihood, no matter the value of $p > 1/2$.

Kemeny Rule (Informally)

So the MLE will be the ranking that has the smallest total number of disagreements with the voters' pairwise orderings.

This is called the Kemeny rule.

Bibliography

Chapter 8 in Brandt et al. (2016): *Handbook of computational social choice.*