Voting in One Dimension

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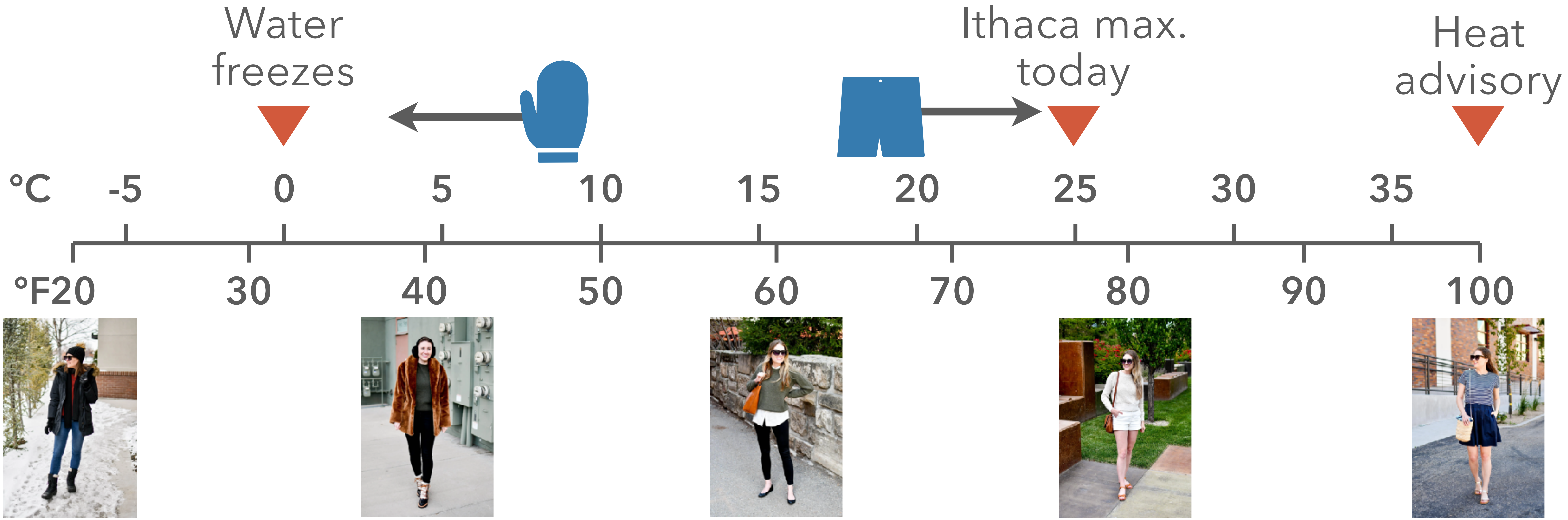
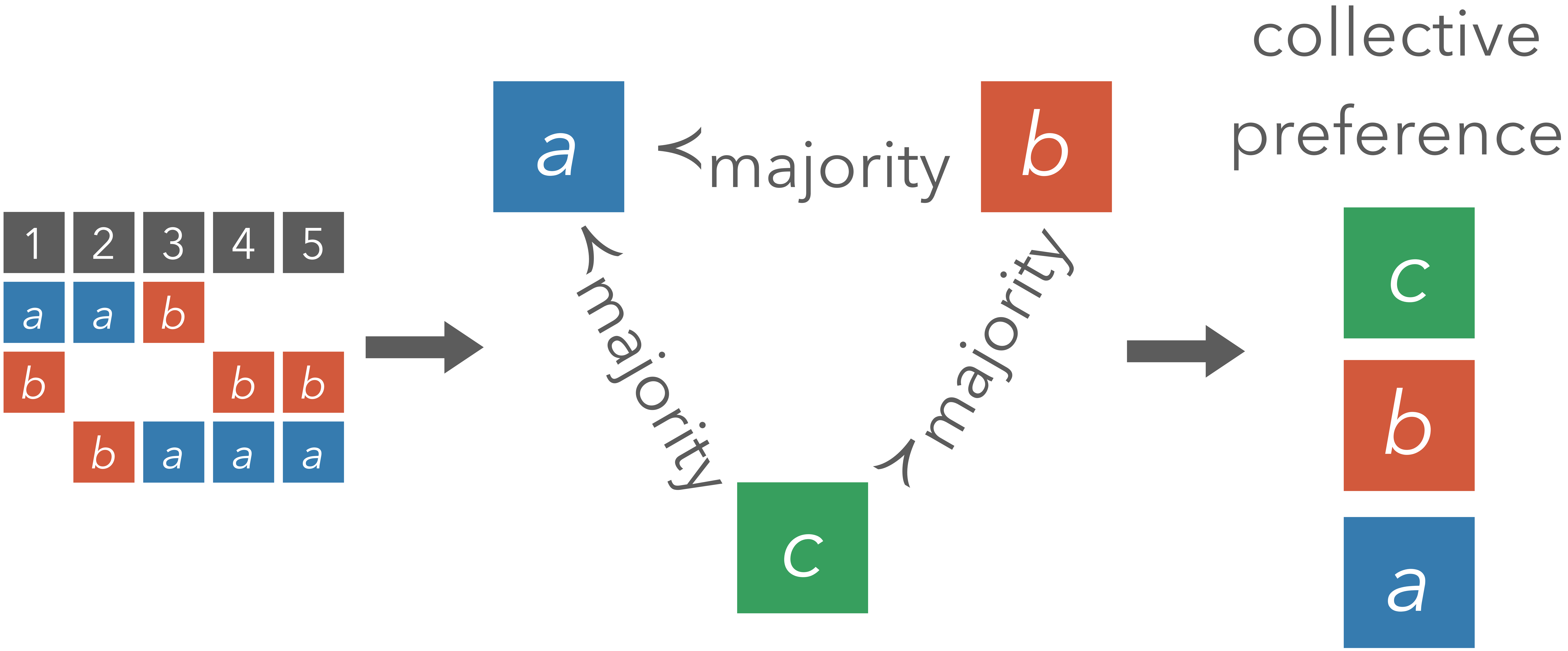
Lecture 4, ORIE 4360/5360, Fall '26

Example for Voting on a Line

Q: What's your favorite outside temperature?

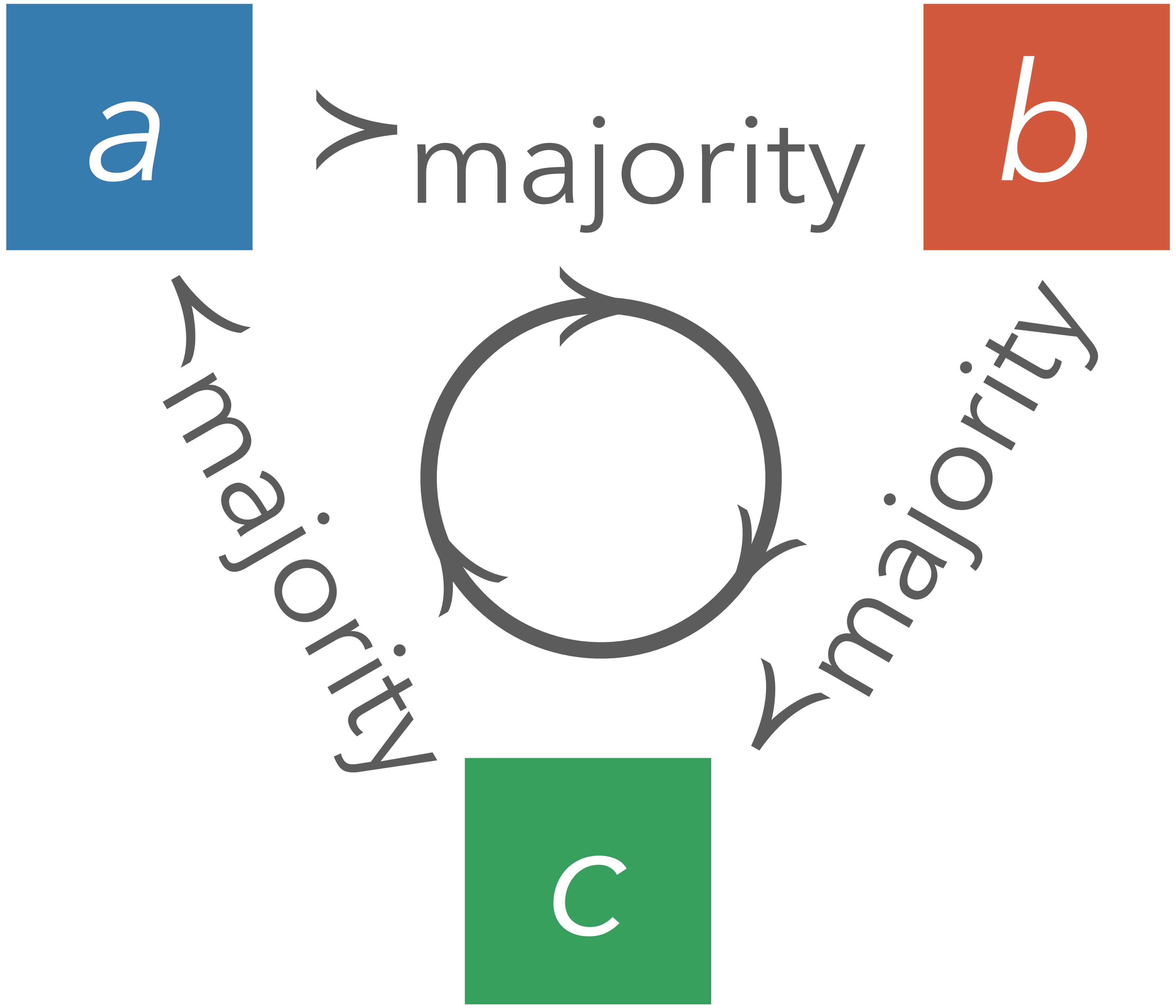


Recap: Natural Idea: Build on Majority



Recap: Condorcet Cycles

1	2	3
a		*b*
b	*a*	
	b	*a*



Recap: Gibbard–Satterthwaite Theorem

Theorem [Gibbard 1973, Satterthwaite 1975]**.** For $m \geq 3$, a social choice function is SP and onto if and only if f is a dictatorship.

Recap: Voting Model

Voters $N = \{1,2,\ldots,n\}$.

Alternatives $M = \{a, b, c, \ldots\}, |M| = m$.

Each voter i has a ranking $\sigma_i \in \mathscr{L}$ over the alternatives; $a \succ_{\sigma_i} b$ means that i prefers a to b.

A preference profile $\boldsymbol{\sigma} \in \mathscr{L}^n$ is a collection of all voters' rankings.

A social choice function is a function $f : \mathscr{L}^n \to M$; a social welfare function is $f : \mathscr{L}^n \to \mathscr{L}$.

Preference Restrictions

Suppose that we know that preferences come from some restricted domain $\mathscr{D} \subseteq \mathscr{L}^n$.

Consider social choice functions $f : \mathscr{D} \to M$, i.e., ones that only take in profiles from $\mathscr{D}$.

Are there natural $\mathscr{D}$ for which we can avoid our negative results?

Condorcet Domains

Theorem. Let n be odd. If all profiles in $\mathscr{D}$ have a Condorcet winner, the social choice function $f_{\text{Cond.}}$ that chooses the Condorcet winner is SP.

Proof. Suppose that $\boldsymbol{\sigma} \in \mathscr{D}$, $\boldsymbol{\sigma}' = (\boldsymbol{\sigma}_{-i}, \sigma_i') \in \mathscr{D}$, $f_{\text{Cond.}}(\boldsymbol{\sigma}) = x$, $f_{\text{Cond.}}(\boldsymbol{\sigma}') = y$, and $y \succ_{\sigma_i} x$.

In $\boldsymbol{\sigma}$, more than $\frac{n}{2}$ voters prefer x over y; i doesn't.

Hence, in $\boldsymbol{\sigma}'$, still more than $\frac{n}{2}$ voters prefer x over y, so y not a Condorcet winner, contradiction. $\square$

Condorcet Domains

Theorem. Let n be odd. If all profiles in $\mathscr{D}$ have a Condorcet winner, the social choice function that chooses the Condorcet winner is SP.

Shows that a lack of Condorcet winners/cycles in majority preferences are really to blame for the vulnerability to manipulation.

Are there settings where we don't expect to encounter them?

Single-Peaked Preferences

Fix some ordering $< \in \mathscr{L}$ over the alternatives.

Voter i's ranking σ_i is single-peaked with respect to $<$ if there is a peak $x_i^* \in M$ such that:

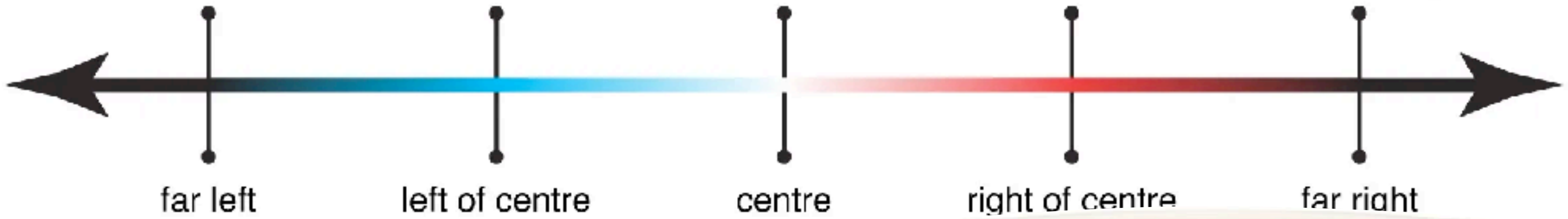
- when $y < z \leq x_i^*$, then $y \prec_{\sigma_i} z$,
- when $x_i^* \leq y < z$, then $y \succ_{\sigma_i} z$.

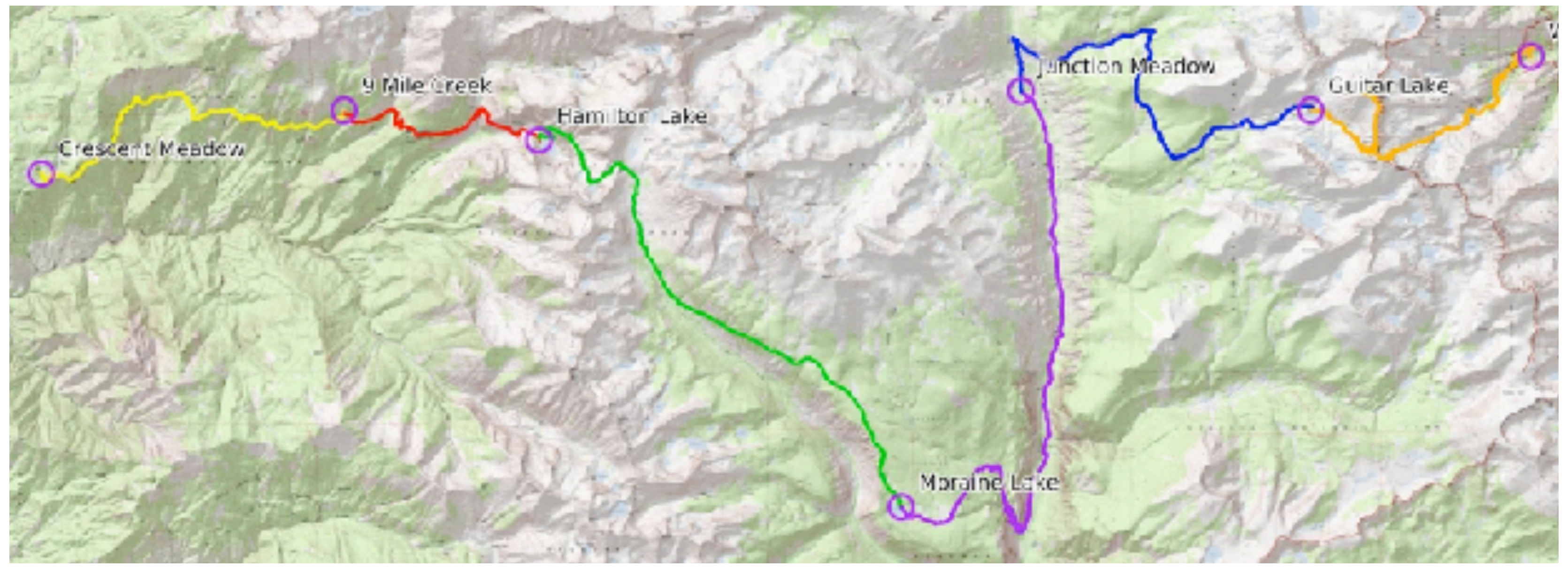
Write $SPL_< \subseteq \mathscr{L}$ for the domain of single-peaked rankings w.r.t. $<$.

σ is single-peaked if it's single-peaked for some $<$.

Single-Peaked Preferences

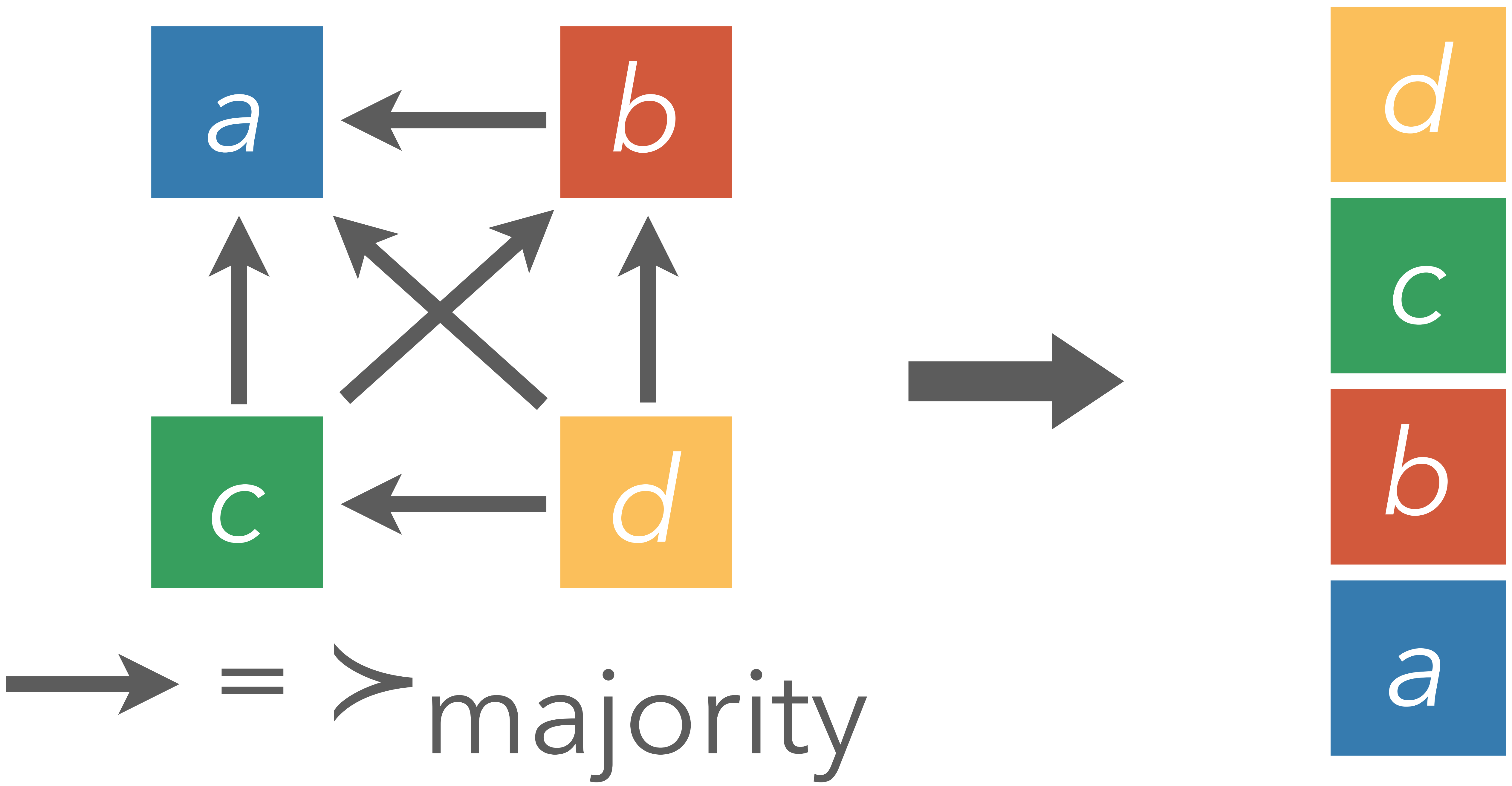
Where might we expect (mostly) single-peaked preferences?





Black's Theorem (Part 1)

Theorem [Black 1948]. For any single-peaked preference profile $\boldsymbol{\sigma} \in SPL^n_<$ and n odd, the majority relationship is transitive.



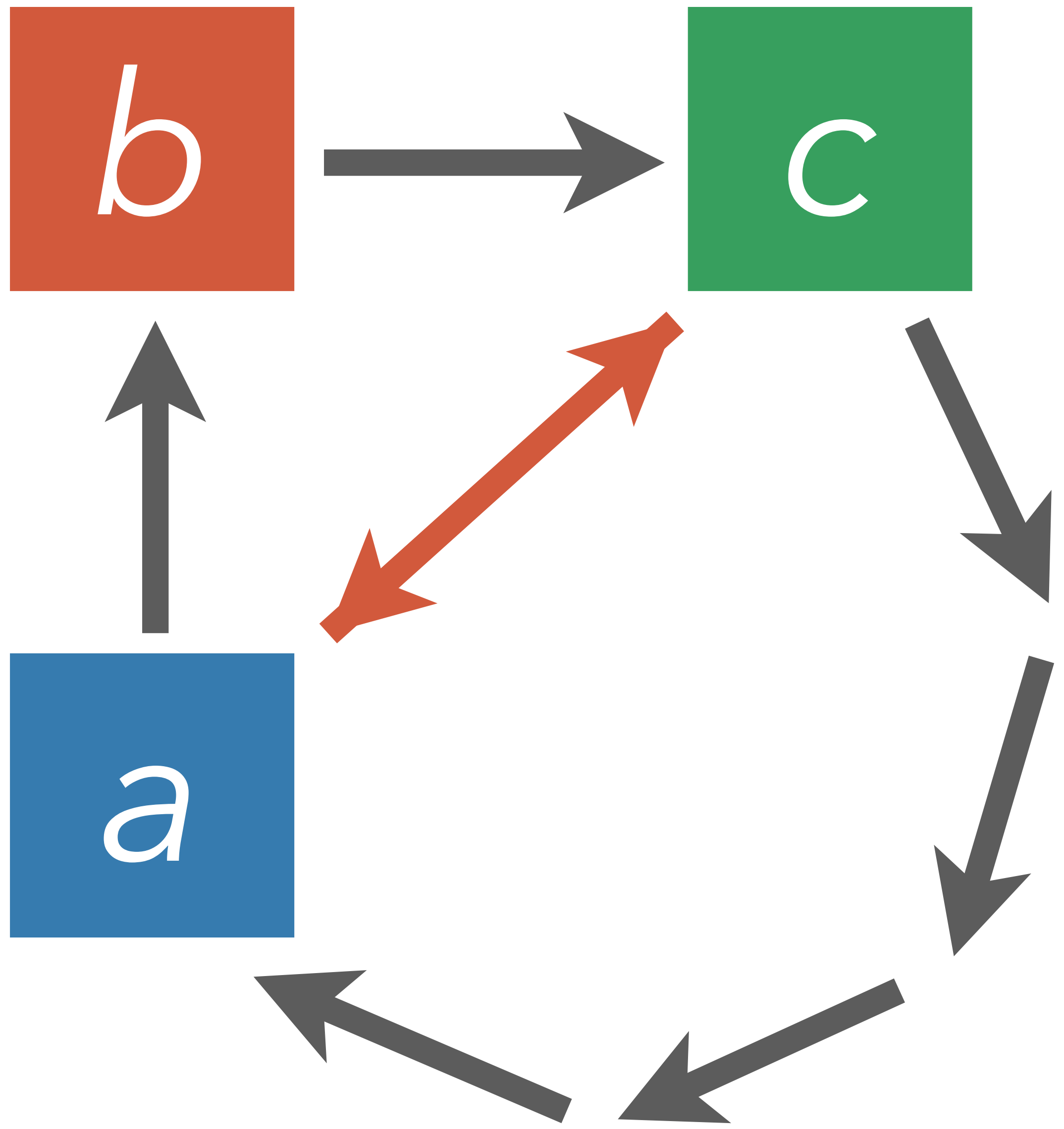
Proof of Black's Theorem (Part 1)

Show by contradiction that $\succ_{\text{majority}}$ is acyclic.

Claim: If there is a cycle, there must be a cycle involving only 3 alternatives.

Indeed, fix 3 consecutive alt's on the shortest cycle. Either

- $a \succ_{\text{majority}} c$: shorter cycle ⚡
- $a \prec_{\text{majority}} c$: found 3-cycle.

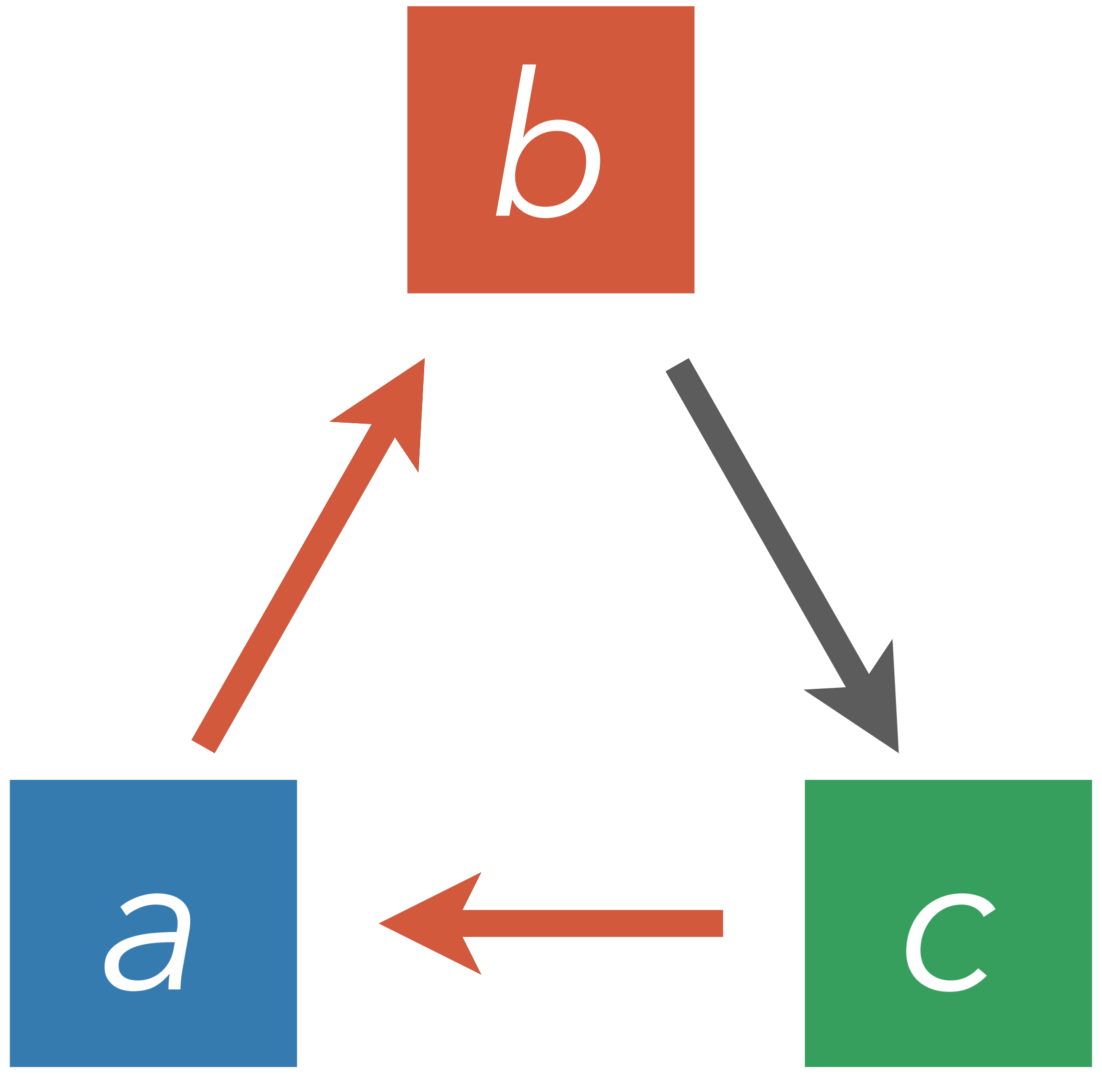


$\longrightarrow \; = \; \succ_{\text{majority}}$

Proof of Black's Theorem (Part 1)

Show by contradiction that $\succ_{\text{majority}}$ is acyclic.

So, fix 3-cycle a, b, c. By symmetry can assume $a < b < c$. Observe that any two strict majorities must have an agent in common. So, for some i, $c \succ_{\sigma_i} a \succ_{\sigma_i} b$. Violates single-peakedness, contradiction. $\square$

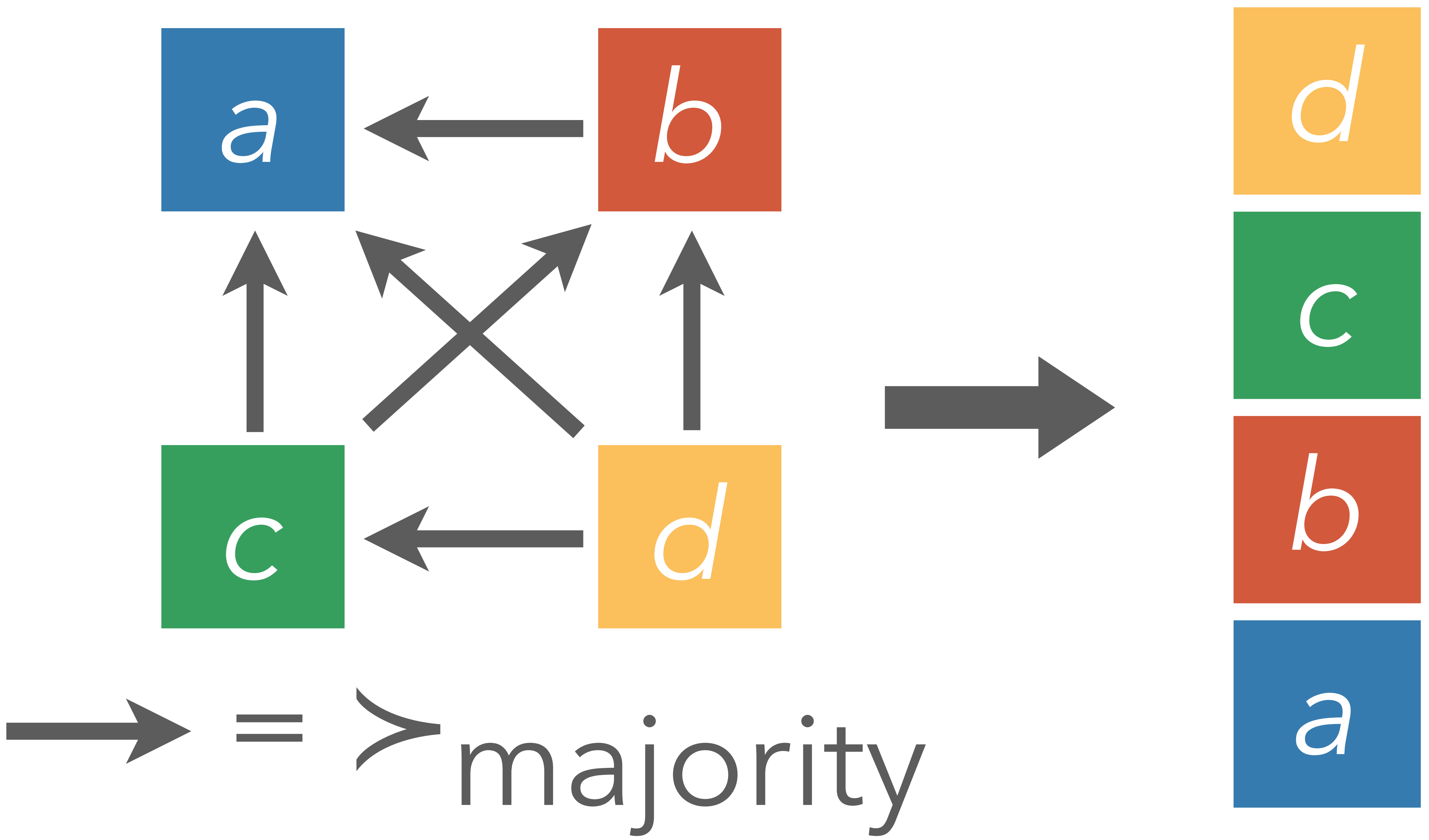


$\rightarrow = \succ_{\text{majority}}$

Black's Theorem (Part 1)

Theorem [Black 1948]. For any single-peaked preference profile $\boldsymbol{\sigma} \in SPL^n_<$ and n odd, the majority relationship is transitive.

Corollary. Single-peaked profiles have a Condorcet winner, and $f_{\text{Cond.}}$ is SP on $SPL^n_<$.

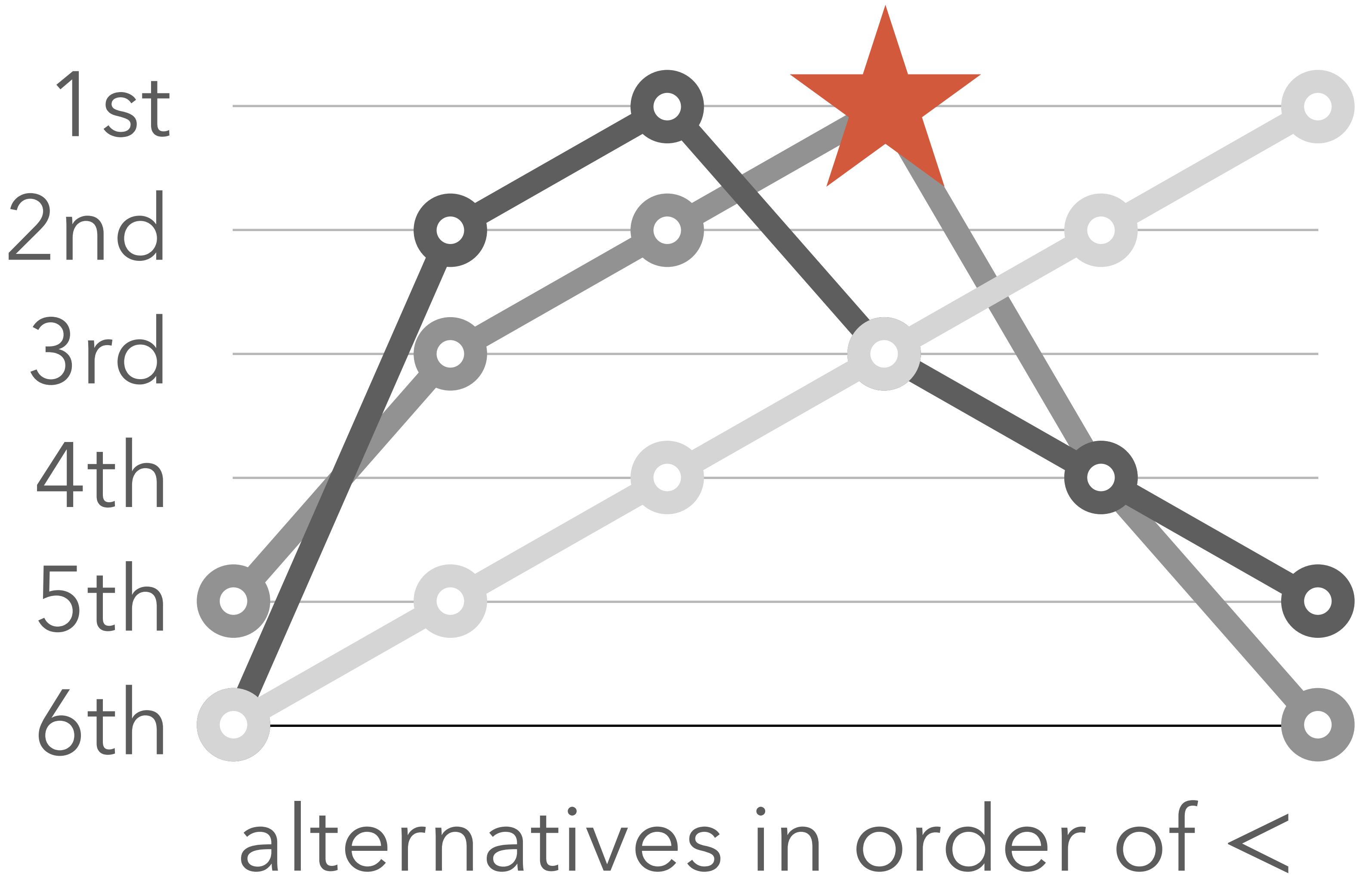


Black's Theorem (Part 2)

Theorem [Black 1948]. For any single-peaked preference profile $\boldsymbol{\sigma} \in SPL_{<}^{n}$ and n odd, the median of the voters' peaks is the Condorcet winner.

Proof. Let m be median, x any other alternative, w.l.o.g., $m < x$.

Then, all voters with peaks $\leq m$ prefer m to x, a majority of voters. $\square$



Other SP Rules for Single-Peakedness

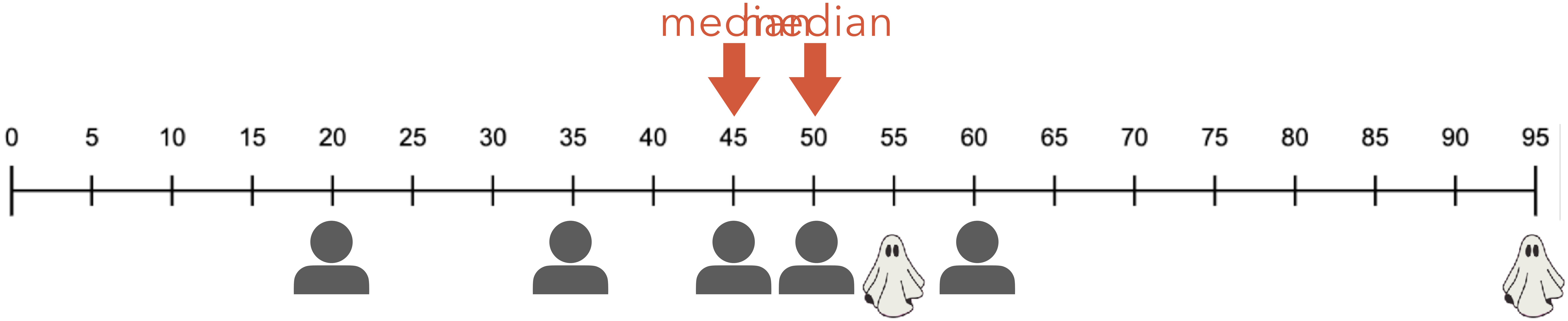
This shows that selecting the median peak is strategyproof for single-peaked preferences.

This social choice function is also clearly anonymous.

Q: Can you think of other social choice functions that are SP and anonymous for single-peaked preferences?

Other SP Rules for Single-Peakedness

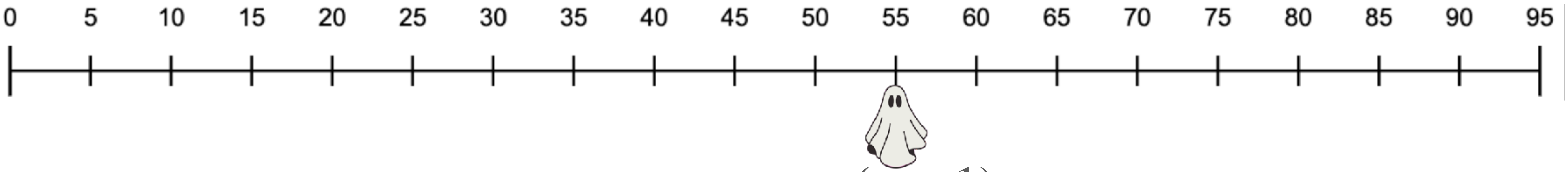
Simple observation: Strategy-proofness and anonymity of selecting the median peak are preserved if we throw in the peaks of fictitious voters (“phantoms”) at some fixed alternatives.



Generalized Median/Phantom Rules

We can do a lot with $n+1$ phantoms:

Constant function (e.g., candidate 55):

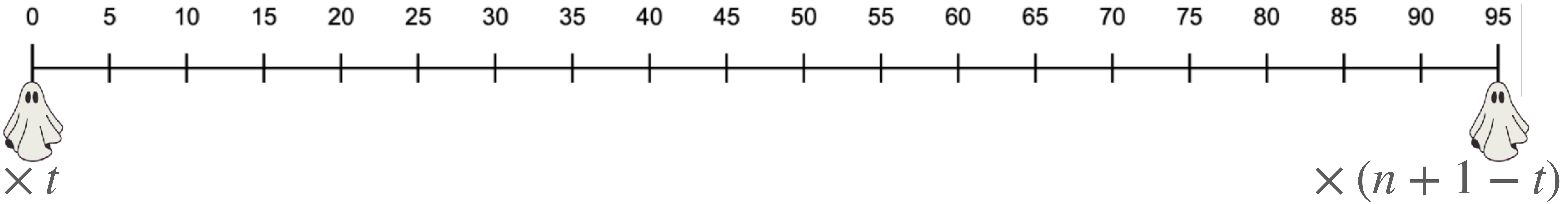


$\times (n+1)$

Generalized Median/Phantom Rules

We can do a lot with $n+1$ phantoms:

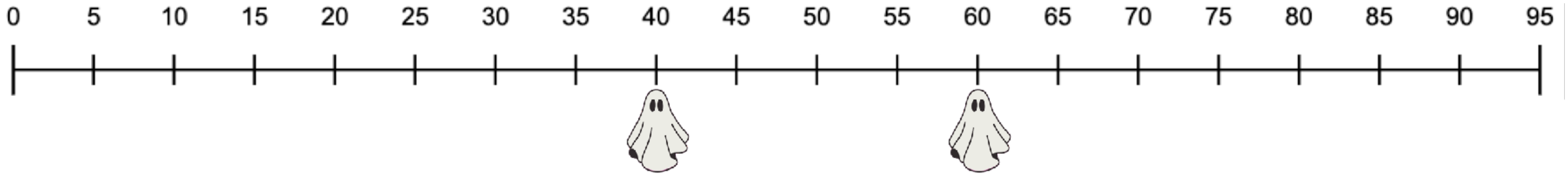
Order statistic of peaks (tth peak from the right):



Generalized Median/Phantom Rules

We can do a lot with $n + 1$ phantoms:

Median within some bounds (e.g., $\min(60, \max(40,$median$)))$:



$\times \frac{n+1}{2}$ $\times \frac{n+1}{2}$

Generalized Median/Phantom Rules

Theorem [Moulin 1980]**.** A social choice function for single-peaked preferences and n odd is SP and anonymous iff it is a generalized median rule with $n + 1$ phantoms.

Bibliography

Elkind, Lackner & Peters (2022): *Preference Restrictions in Computational Social Choice*.

Moulin: *On Strategy-Proofness and Single Peakedness*. Public Choice, 1980.