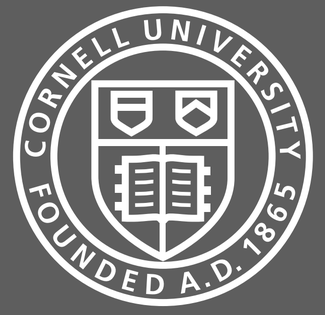
The "Impossibility" of Social Choice

Paul Gölz

Lecture 2, ORIE 4360/5360, Fall '26



Recap: Case of Two Alternatives

Theorem [May 1952]. Suppose that $m = 2$ and n is odd. Then, $f : \mathscr{L}^n \to A$ is anonymous, neutral, and monotone if and only if it is majority.

51%: $a \succ b$

49%: $b \succ a$

Tempting to say:

"The people" prefer a over b.

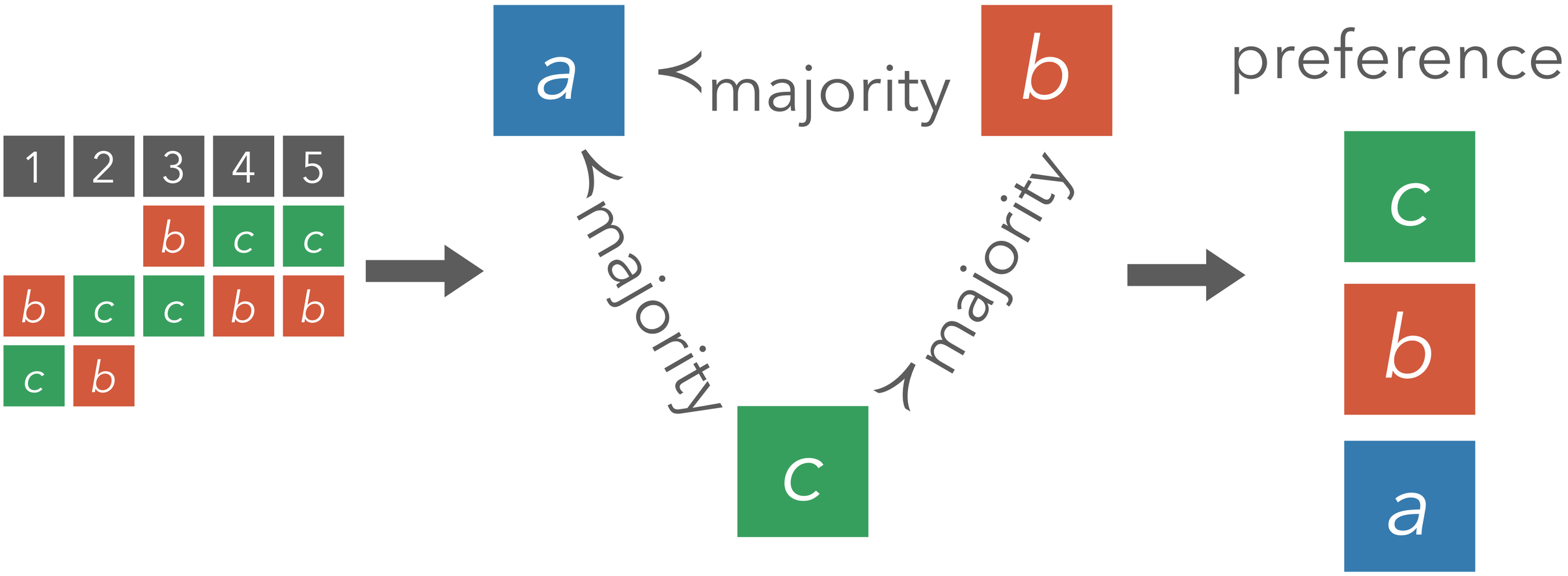
Implicit idea:

"The people" have preferences that look like an individual's, which can be revealed by voting.

Illustration of Hobbes' *Leviathan* (1651)

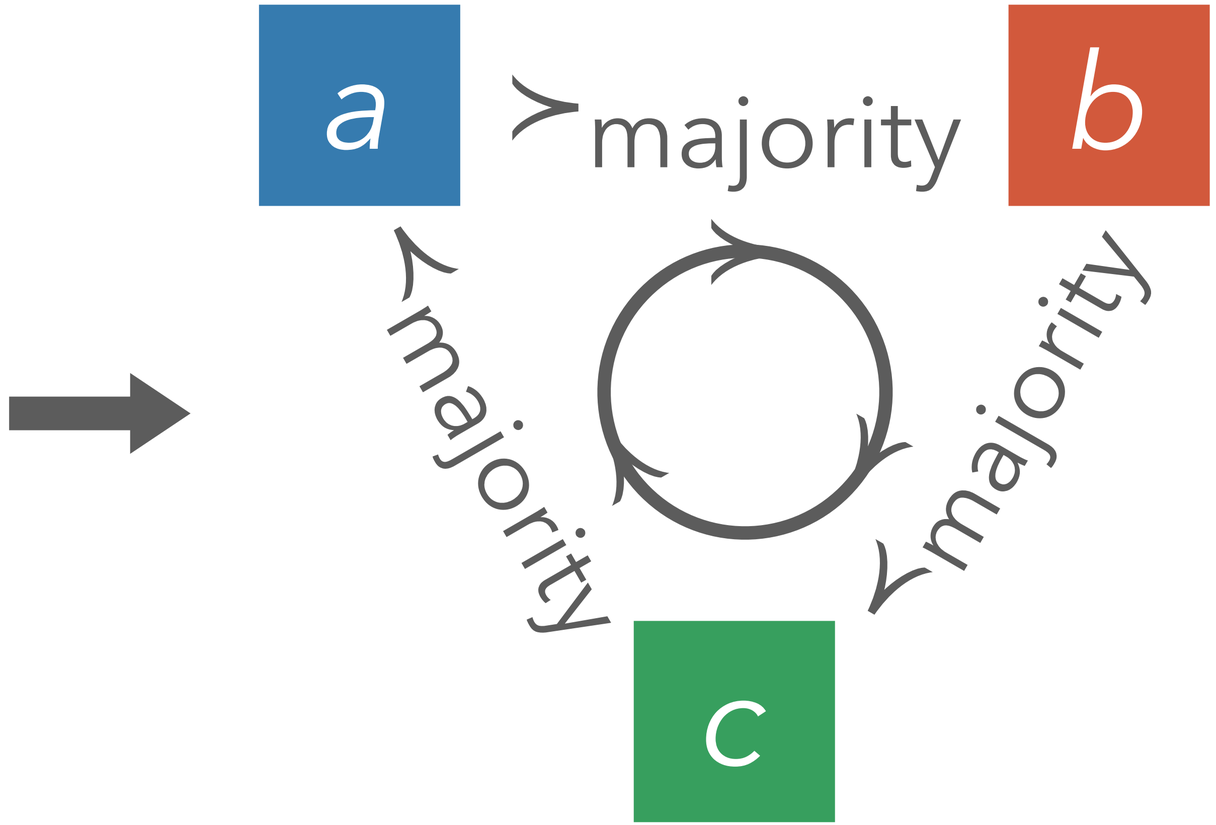
Onward to $m \geq 3$!

Natural Idea: Build on Majority



Condorcet Cycles

1	2	3
a		*b*
b	*a*	
	b	*a*



Condorcet Winners & Consistency

An alternative x is called a Condorcet winner if, for each other alternative y, at least half of the agents prefer x over y.

Nicolas de Condorcet (1743–1794)

Q: Can there ever be more than one Condorcet winner?

Answer: Yes, for example when $m = 2$ and the alternatives are tied. Ties are necessary for this.

Condorcet Winners & Consistency

Nicolas de Condorcet (1743–1794)

An alternative x is called a Condorcet winner if, for each other alternative y, at least half of the agents prefer x over y.

A social choice function is called Condorcet consistent if it selects a Condorcet winner whenever one exists for the given profile.

Plurality & STV Aren't Cond. Consistent

1	2	3	4	5
a	*a*	*b*	*c*	*d*
d	*d*	*d*	*d*	*c*
b	*c*	*c*	*b*	*b*
c	*b*	*a*	*a*	*a*

In example from last time, d is the unique Condorcet winner.

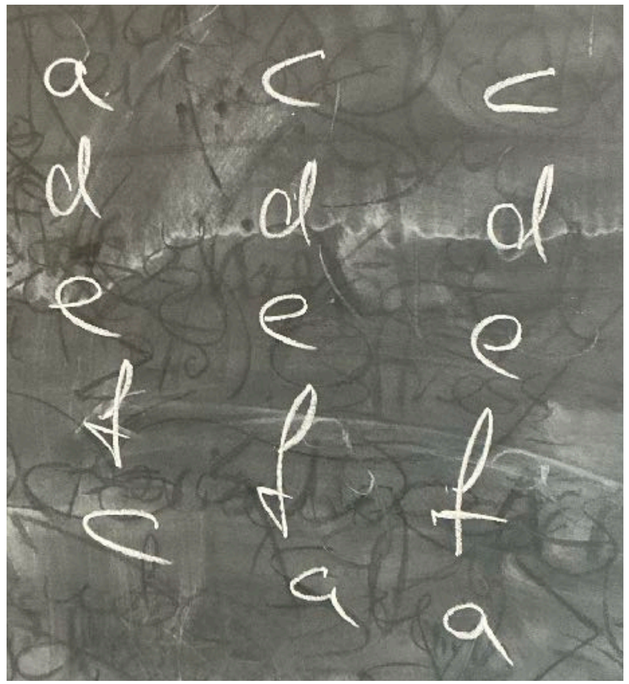
But a is the winner for plurality and c for STV*.

*For one way of breaking ties, can also find counterexamples without ties.

Question

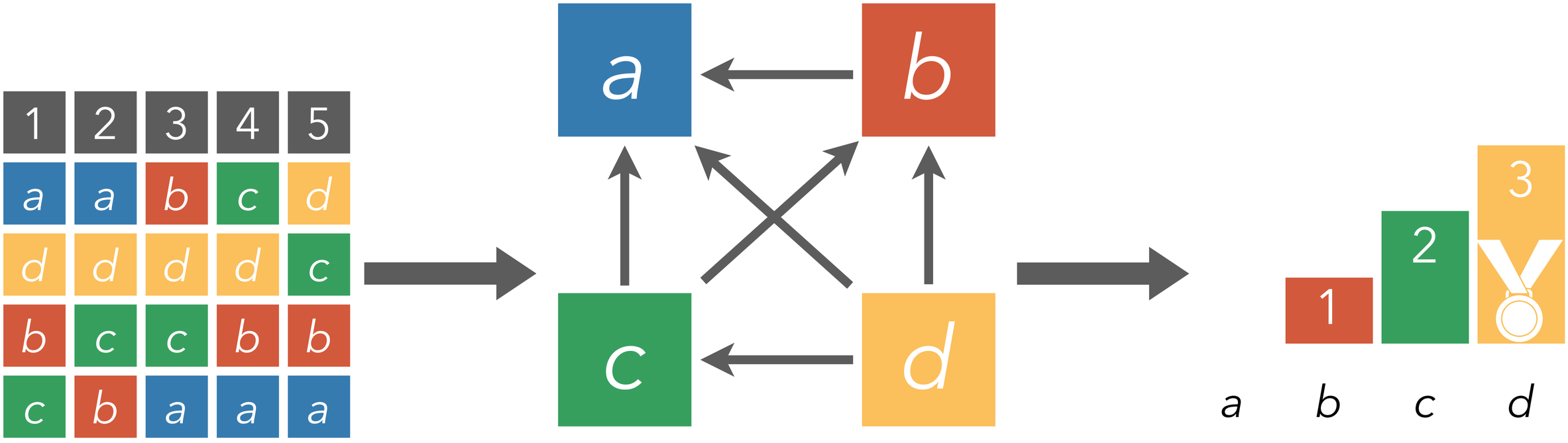
Q: Is Borda Condorcet consistent? Explain why, or find a counter-example!

Answer: No. Counter-example from class:



Copeland Voting Rule

An alternative gets 1 point for each head-to-head election it wins (½ point if tied). Choose alternative with highest score.



$\rightarrow = \succ_{\text{majority}}$

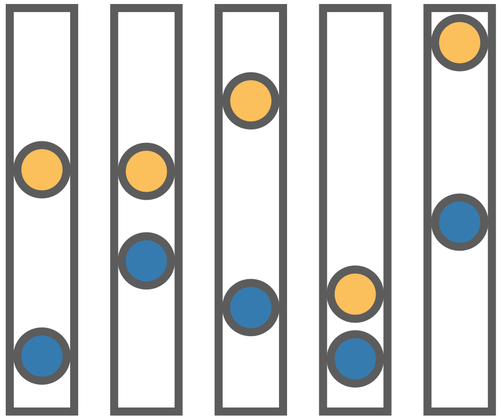
Arrow's Impossibility Theorem

Pairwise majority doesn't recreate "rational" collective preferences.

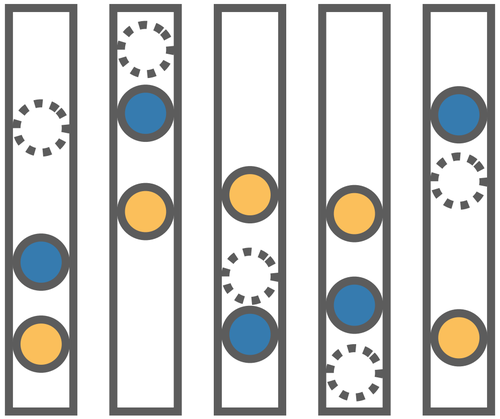
Can some social welfare function save the idea of creating rational collective preferences from votes?

Kenneth Arrow
(1921–2017)

Arrow's Axioms

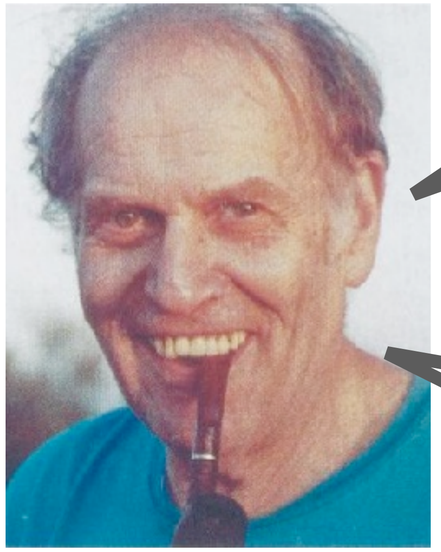


Unanimity: If all voters rank x above y, so does the social welfare function.

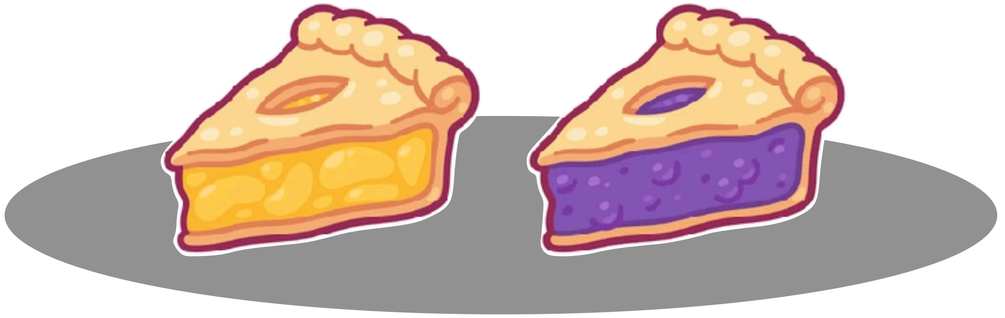


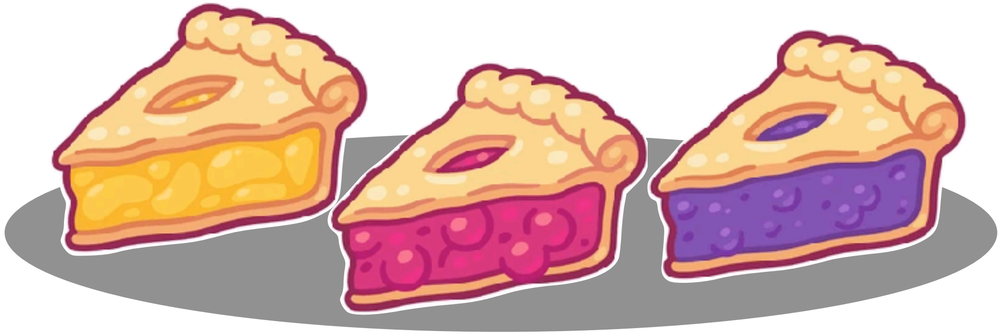
Independence of irrelevant alternatives (IIA): The social ranking over x and y depends only on each voter's ranking restricted to x and y.

Why IIA?



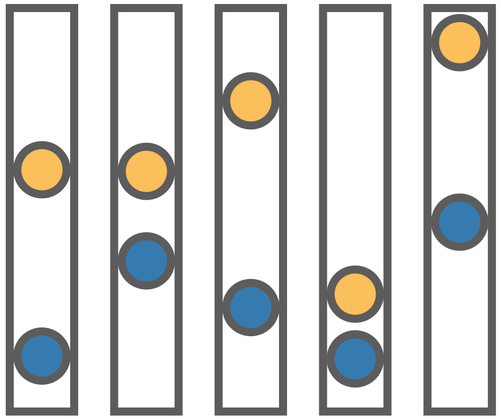
Sydney Morgenbesser
(1921–2004)



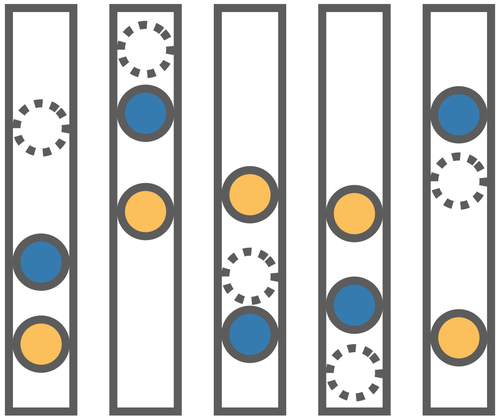


Anecdote after Poundstone: *Gaming the Vote* (2008).

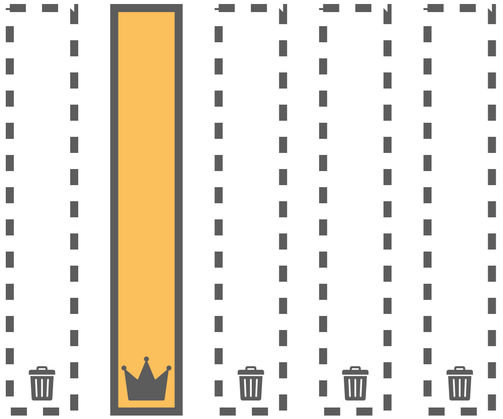
Arrow's Axioms



Unanimity: If all voters rank x above y, so does the social welfare function.



Independence of irrelevant alternatives (IIA): The social ranking over x and y depends only on each voter's ranking restricted to x and y.



Nondictatorship: There isn't any one voter such that the social welfare function just returns that voter's ranking, ignoring all others.

Arrow's Impossibility Theorem

Theorem [Arrow 1950]. For $m \geq 3$, no $f : \mathscr{L}^n \to \mathscr{L}$ satisfies unanimity, IIA, and nondictatorship.

Or, rephrased: only dictatorships satisfy unanimity and IIA.

Q: For $m \geq 3$, do any rules satisfy unanimity and nondictorship? IIA and nondictatorship?

Extremal Lemma

Lemma. If, in given profile, every voter ranks *x* first or last, a unanimous and IIA social welfare function must rank *x* first or last.

[Proof on blackboard, after Geanakoplos (2005) reference on last slide.]

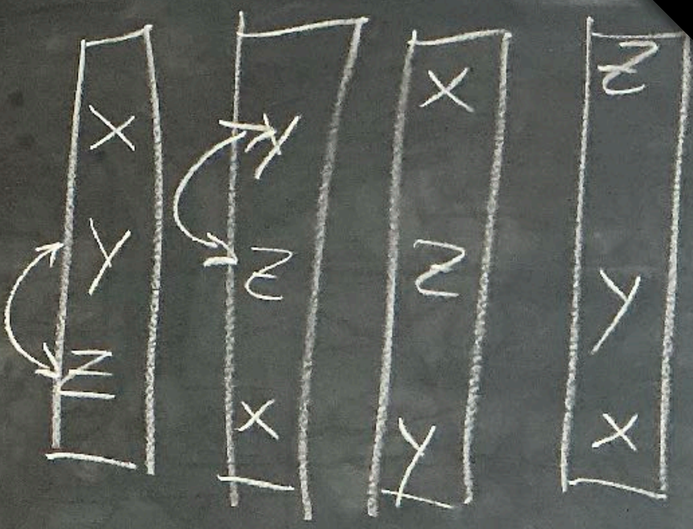
Proof of extremal lemma.

By contradiction: Fix f, σ. Suppose for contradiction $y \succ_{f(\sigma)} x \succ_{f(\sigma)} z$.

We construct σ by flipping y and z for all voters who prefer y over z.

By unanimity, $z \succeq_{f(\sigma')} y$.

Since voters' preferences between x and y and x and z haven't changed, by IIA: $y \succ_{f(\sigma')} x \succ_{f(\sigma')} z$. Contradiction. □



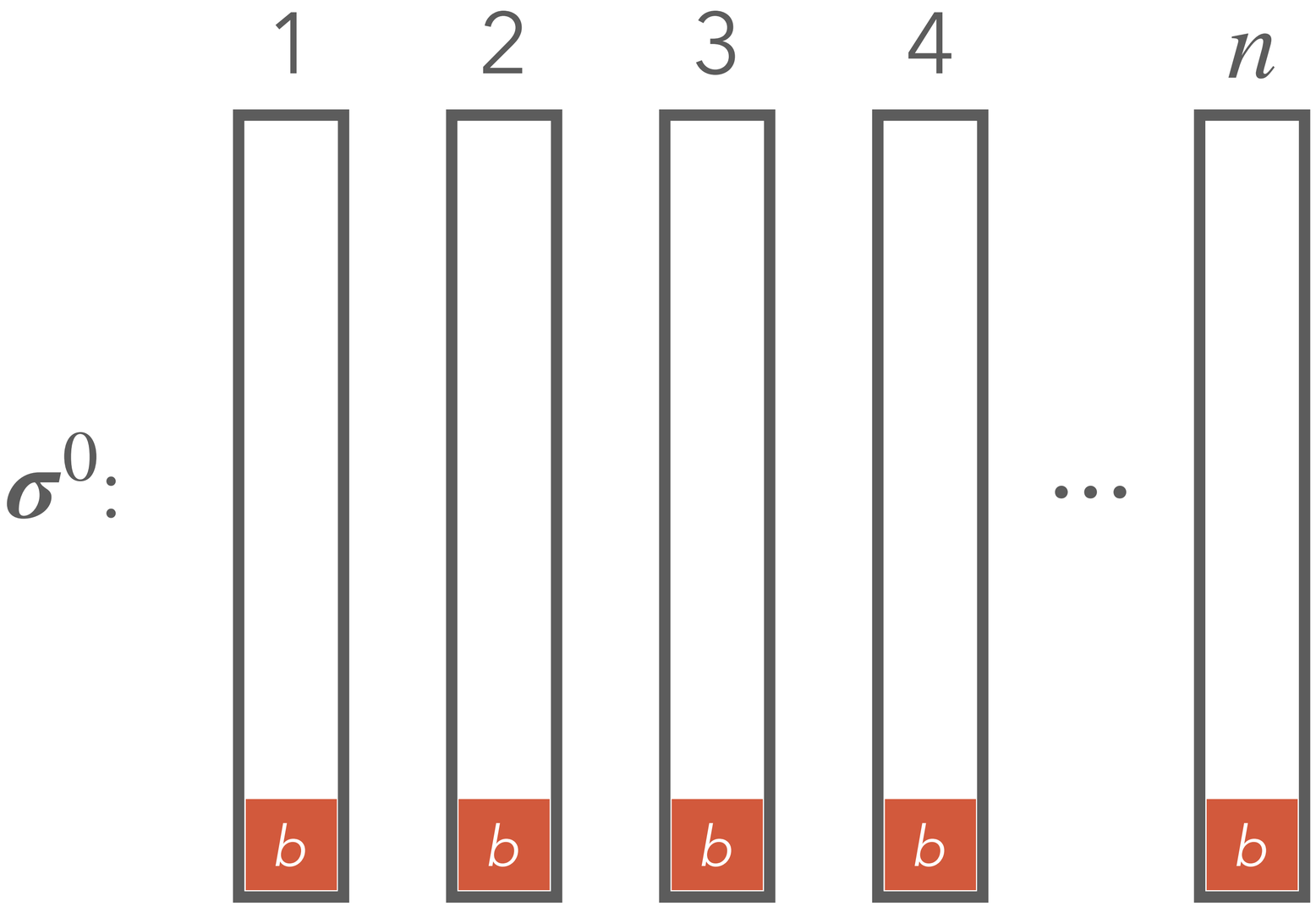
Proof Overview

Now, proof in 4 steps.

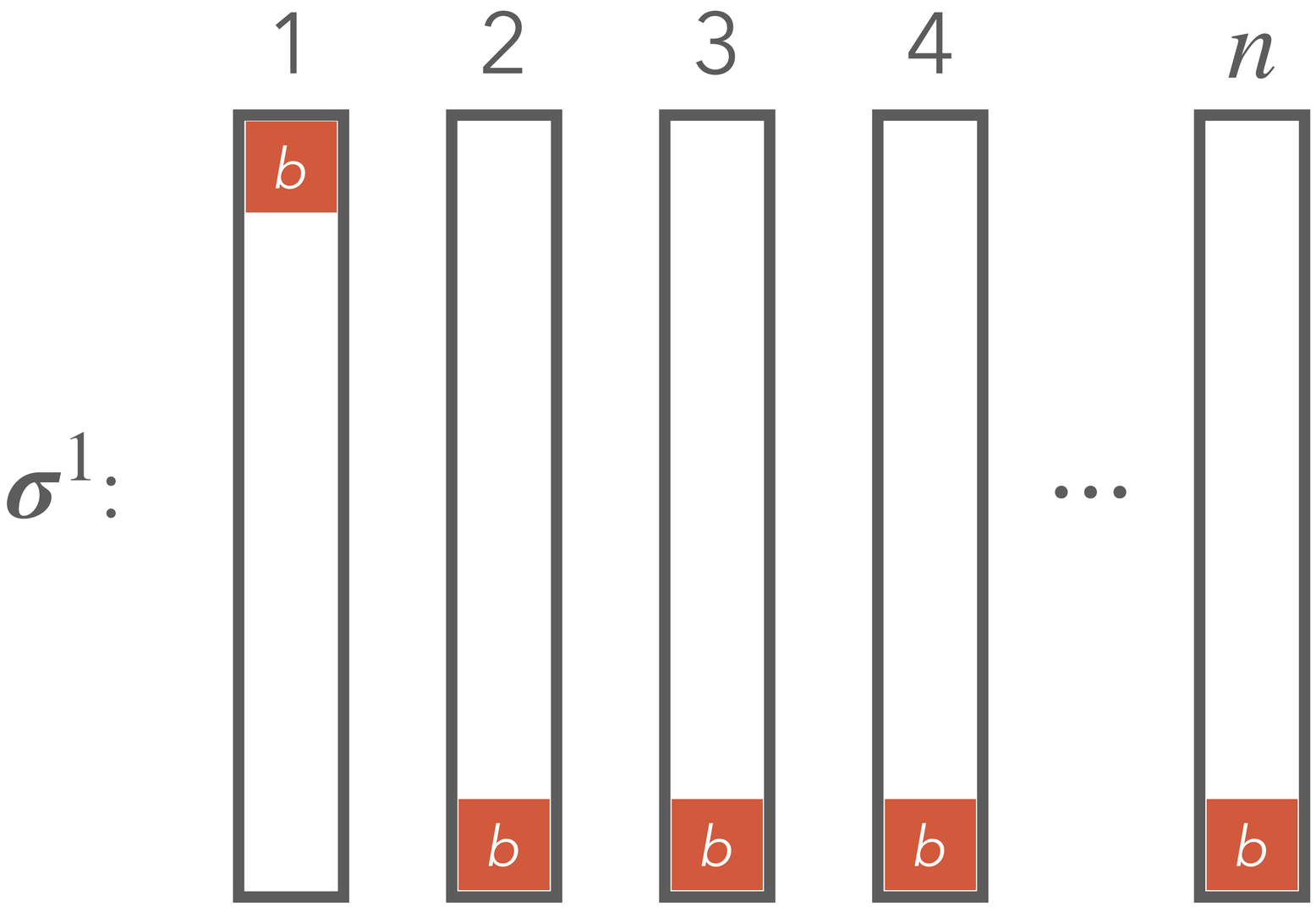
Step 1: Roughly, find profile where one voter can make alternative b ranked first or last in output.

Fix an arbitrary profile $\boldsymbol{\sigma}^0$ in which all voters rank b last.

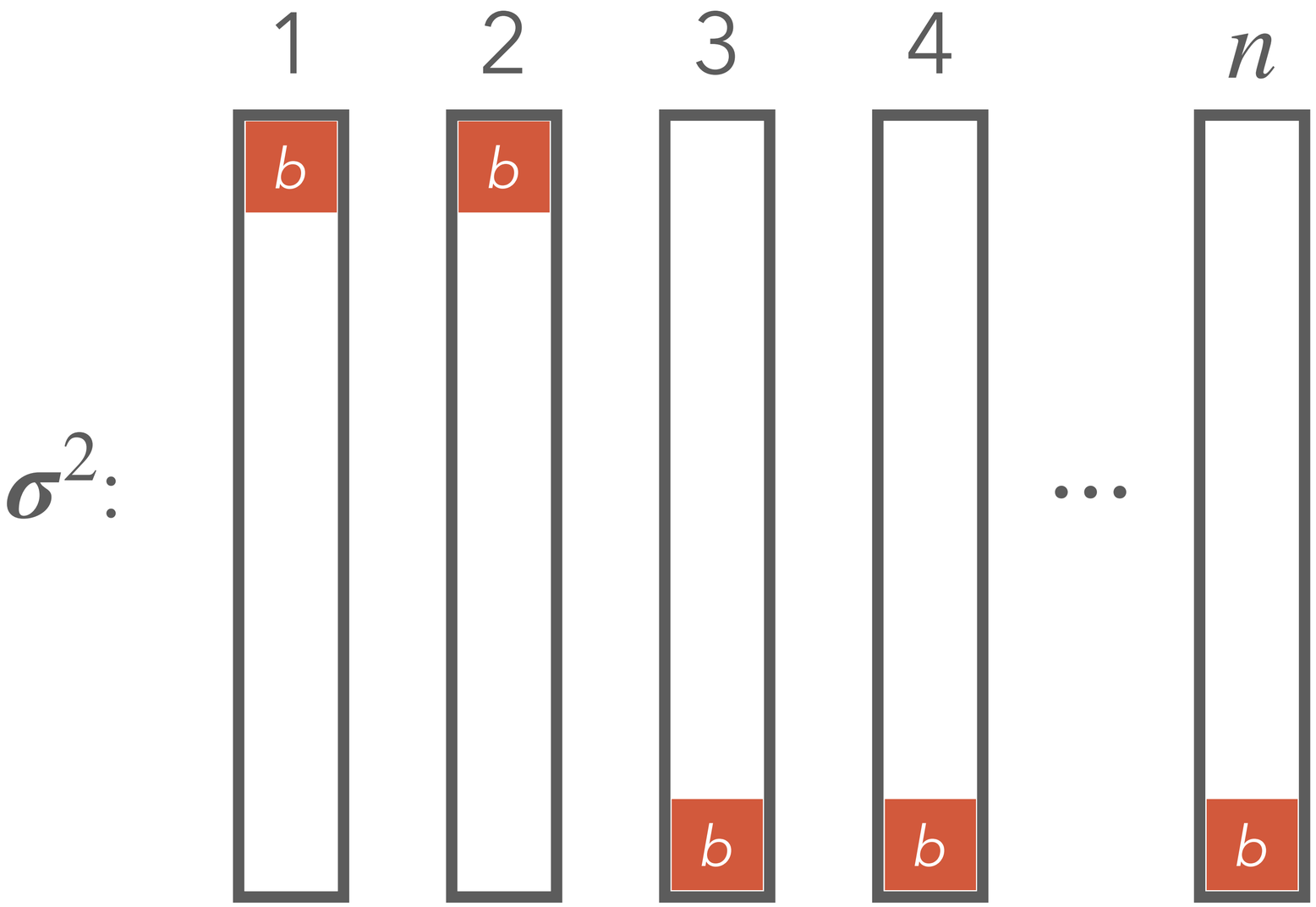
Proof Step 1



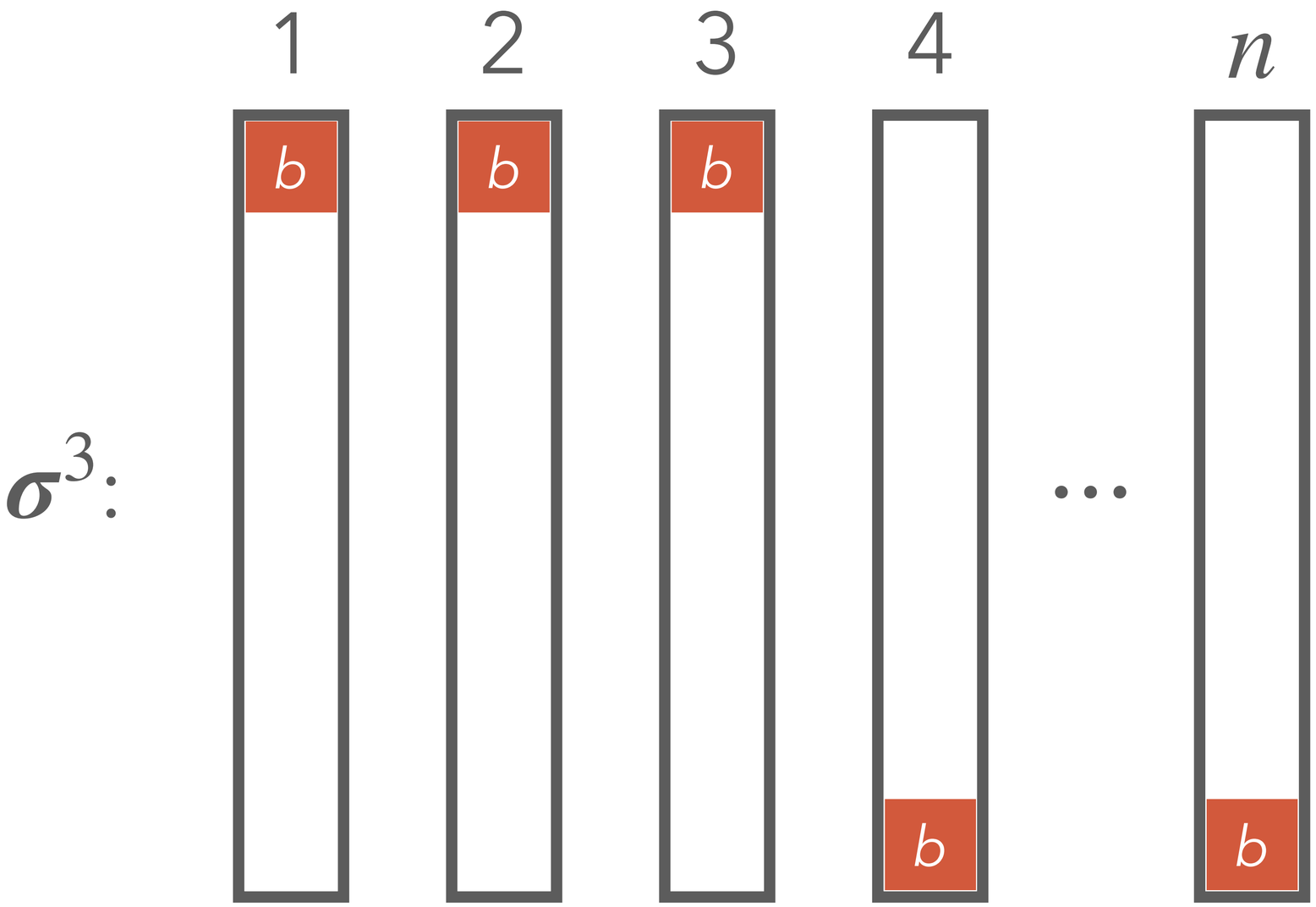
Proof Step 1



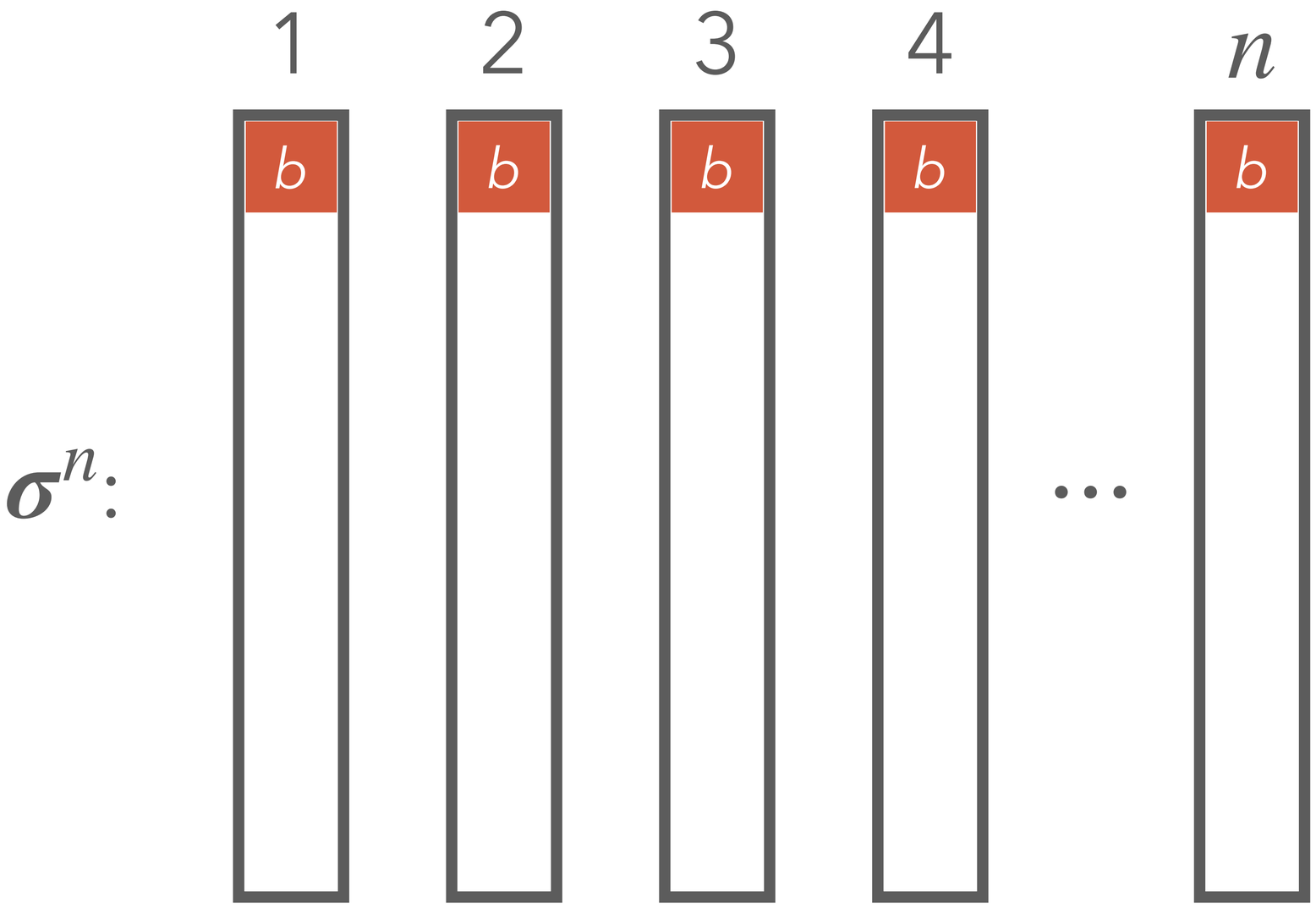
Proof Step 1



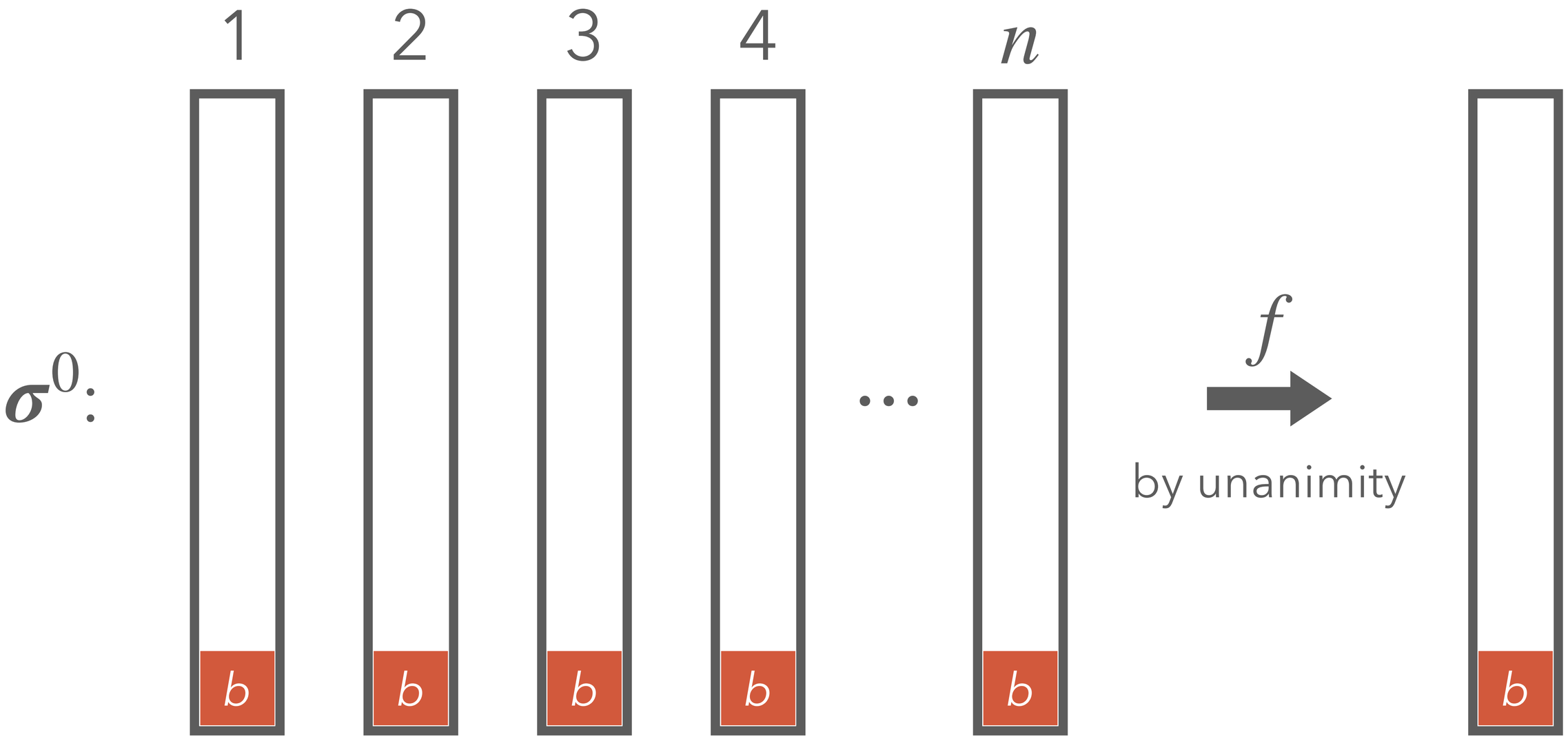
Proof Step 1



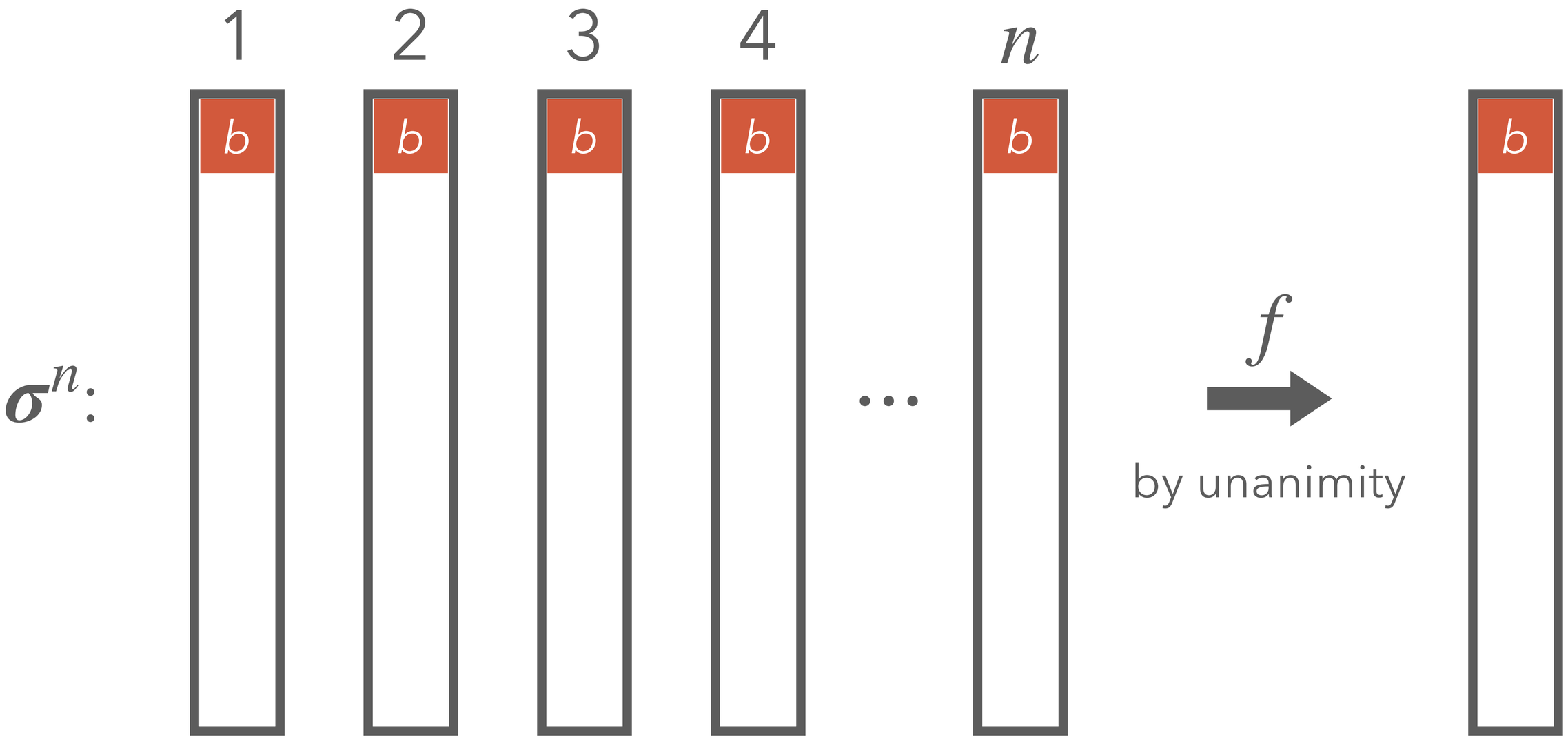
Proof Step 1



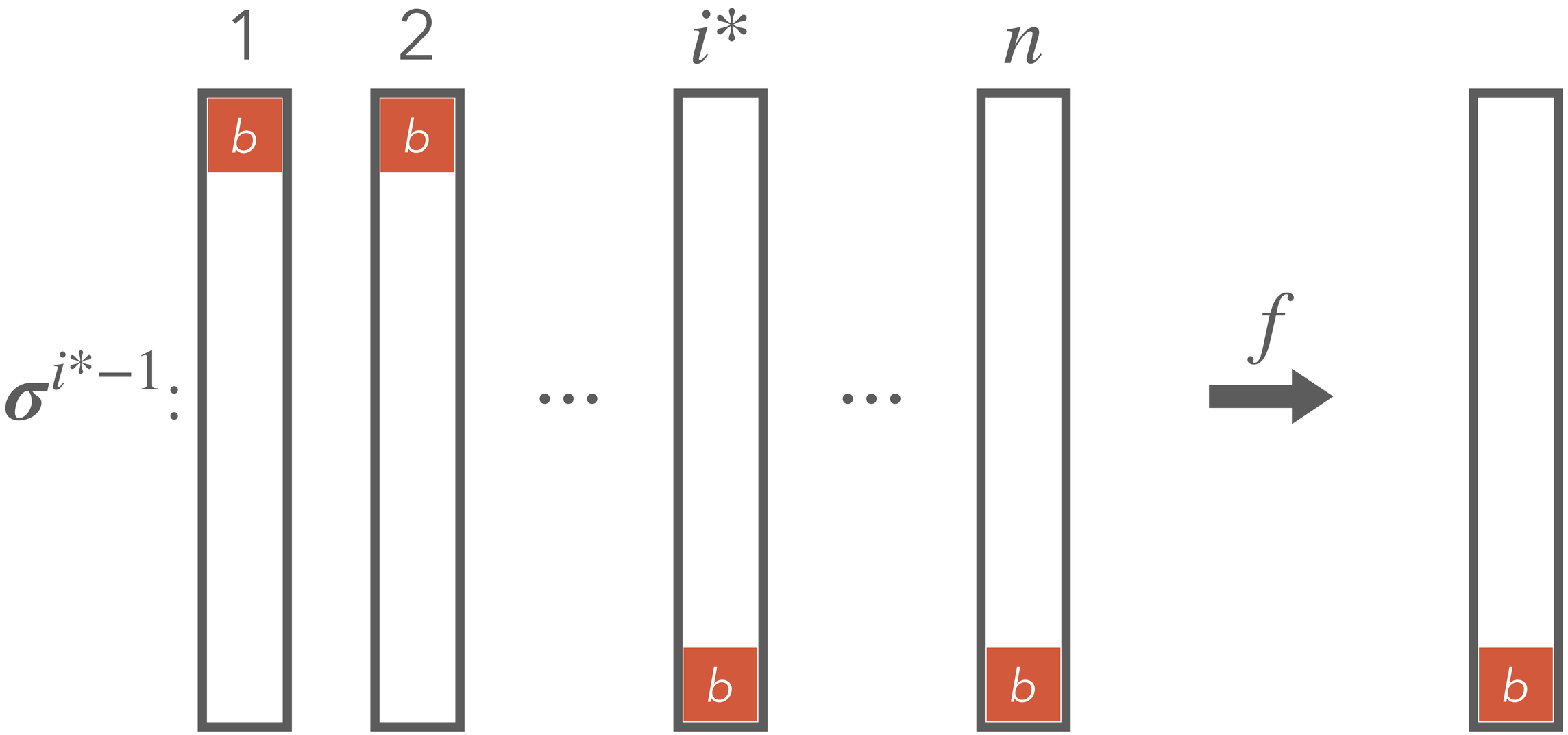
Proof Step 1



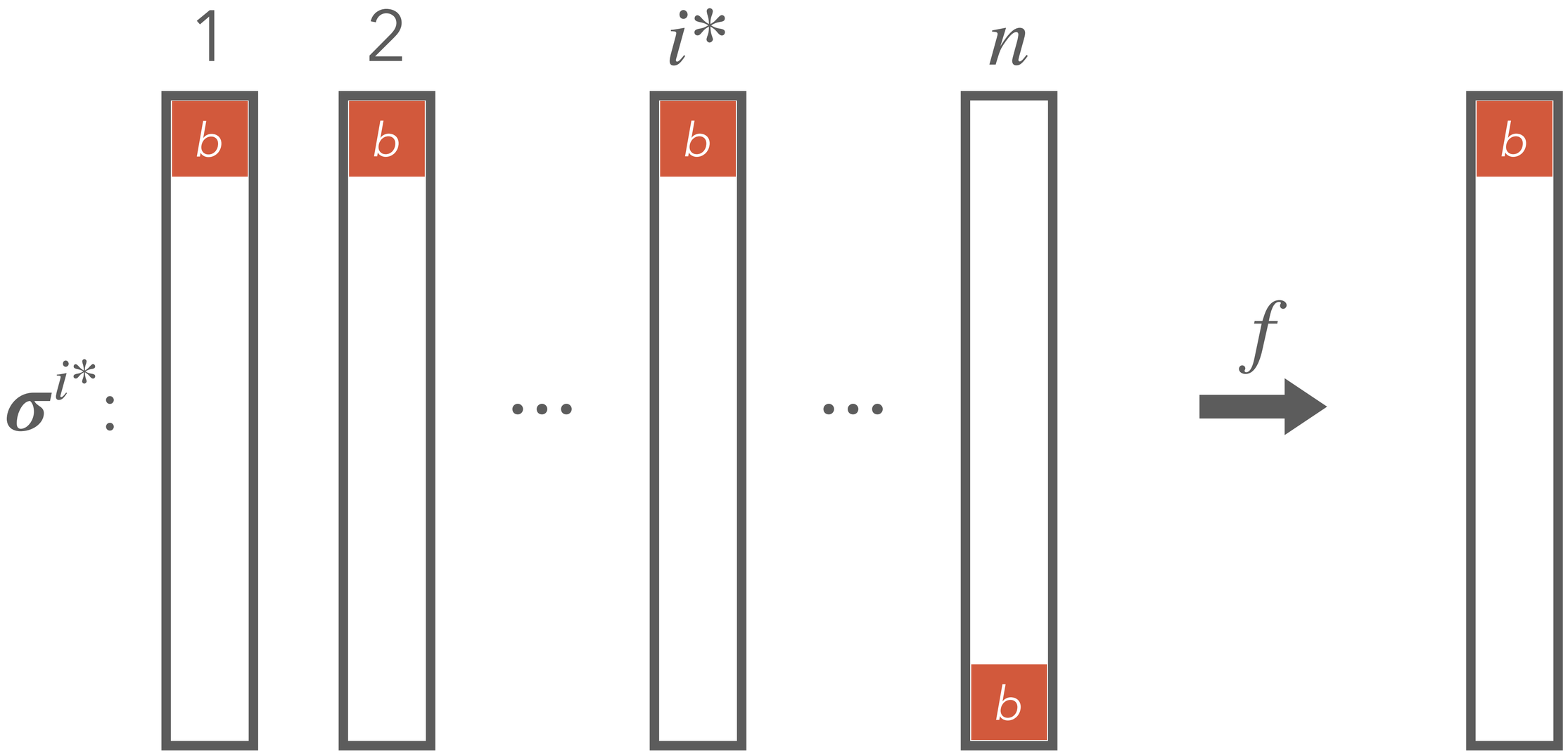
Proof Step 1



Proof Step 1



Proof Step 1



Step 2

Step 1: Found profiles $\boldsymbol{\sigma}^{i^*-1}$ and $\boldsymbol{\sigma}^{i^*}$ where agent i^* moving b from bottom to top causes b to move from bottom to top in social ranking.

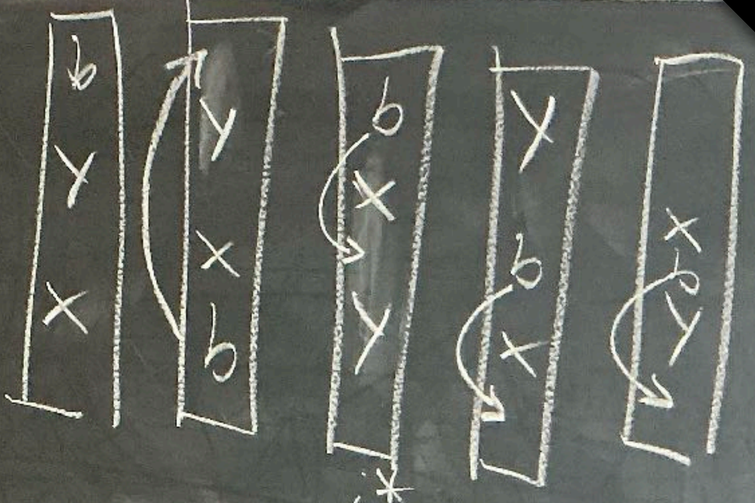
Step 2: i^* is a dictator for all pairs of alternatives x, y not involving b.

Three voters preference, but $f(\sigma)$

Proof of step 2:

Fix any σ and x, y such that $b \notin \{x, y\}$.

WTS: if $x \succ y$, then $x \succ_{f(\sigma)} y$.



Build profile σ^{i^*} from σ by moving b to the top for all $i < i^*$, to the bottom for all $i > i^*$, and between x and y for i^*.

Just considering $\{x, b\}$, this profile looks like σ^{i^*-1}.

Because $x \succ_{f(\sigma^{i^*-1})} b$, by IIA, $x \succ_{f(\sigma^{i^*})} b$.

Same reasoning for y and b, σ^{i^*}: $b \succ_{f(\sigma^{i^*})} y$.

$\left.\begin{array}{l} x \succ_{f(\sigma^{i^*})} b \\ b \succ_{f(\sigma^{i^*})} y \end{array}\right\}$ By IIA, $x \succ_{f(\sigma)} y$.

$x \succ_{f(\sigma)} y$. □

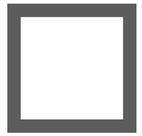
Step 3

For alternative a, the reasoning from previous steps gives that some agent i^{**} is a dictator for all pairs not involving a, e.g., for b, c.

We know that, in $\boldsymbol{\sigma}^{i^*-1}, \boldsymbol{\sigma}^{i^*}$, i^* affected the output ranking of b, c. Hence, $i^* = i^{**}$.

Repeat for c.

It follows that i^* is a dictator.



Arrow's Impossibility Theorem

Theorem [Arrow 1950]. For $m \geq 3$, no $f : \mathscr{L}^n \to \mathscr{L}$ satisfies unanimity, IIA, and nondictatorship.

Impossibility persists if can return rankings with ties, or for social choice functions (see last slide).

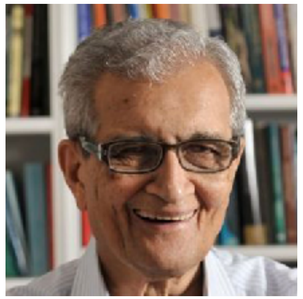
Reactions to Arrow's Theorem

"The search of the great minds of recorded history for the perfect democracy, it turns out, is the search for a chimera, for a logical self-contradiction. ...
Now scholars all over the world-in mathematics, politics, philosophy and economics are trying to salvage what can be salvaged from Arrow's devastating discovery."

Paul Samuelson
(1915–2009)

Reactions to Arrow's Theorem

"The big bang that characterized the beginning [of modern social choice theory] took the form of an 'impossibility theorem'."



Amartya Sen
(*1933)

Arrow's Axioms

Unanimity: If all voters rank x above y, so does the social welfare function.

Independence of irrelevant alternatives (IIA): The social ranking over x and y depends only on each voter's ranking restricted to x and y.

Nondictatorship: There isn't any one voter such that the social welfare function just returns that voter's ranking, ignoring all others.

Q: Which of Arrow's axioms do you feel we could best do without?

Bibliography

Geanakoplos: *Three brief proofs of Arrow's Impossibility Theorem*. J. Economic Theory, 2005.

For Arrow's Theorem for social choice functions, see Theorem 1.3.1 in

Taylor (2005): Social choice and the mathematics of manipulation.