City Sampling for Citizens' Assemblies

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Abstract

In *citizens' assemblies*, a group of constituents is randomly selected to weigh in on policy issues. We study a two-stage sampling problem faced by practitioners in countries such as Germany, in which constituents' contact information is stored at a municipal level. As a result, practitioners can only select constituents from a bounded number of cities ex post, while ensuring equal selection probability for constituents ex ante.

We develop several algorithms for this problem. Although minimizing the number of contacted cities is NP-hard, we provide a pseudo-polynomial time algorithm and an additive 1-approximation, both based on separation oracles for a linear programming formulation. Recognizing that practical objectives go beyond minimizing city count, we further introduce a simple and more interpretable greedy algorithm, which additionally satisfies an ex-post monotonicity property and achieves an additive 2-approximation. Finally, we explore a notion of ex-post proportionality, for which we propose two practical algorithms: an optimal algorithm based on column generation and integer linear programming and a simple heuristic creating particularly transparent distributions. We evaluate these algorithms on data from Germany, and plan to deploy them in cooperation with a leading nonprofit organization in this space.

1 Introduction

Citizens' assemblies are an emerging form of democratic participation, in which a random sample of constituents formulate policy recommendations. The random selection of assembly members, called *sortition*, gives each person an equal chance to participate and ensures that the assembly forms a cross section of the population. Citizens' assemblies have been increasing in frequency [OECD, 2020]. National-level examples include assemblies on same-sex marriage, abortion, and gender equality in Ireland [Courant, 2021] and German assemblies on the country's global role [Mehr Demokratie, 2021], nutrition [Deutscher Bundestag, 2024], and disinformation [Bertelsmann, 2024].

In practice, the sortition proceeds in two stages: first, a large number of random constituents are invited by mail; second, the members of the assembly are selected among those invited who volunteered to participate. Most algorithmic work on citizens' assemblies focuses on the second stage [Flanigan et al., 2020, 2021a, 2024, 2021b, Baharav and Flanigan, 2024].

This work, instead, studies a practical problem arising in the first sampling stage in certain countries. Sampling constituents with equal probability is straight-forward in countries with a central population register such as the Nordic countries [Scherpenzeel et al., 2017]. The sampling process is also simple in countries like the UK and US where no register exists and assembly organizers use postal lists to invite random households, though these lists under-represent "rural areas, ..., Hispanic households, non-English-speaking households" among others [Kalton et al., 2014].

The first sampling stage is more complex in countries such as Germany and Italy, where population registers are kept by municipalities. Since these municipalities must be individually petitioned for sampling access in a burdensome process [Stadtmüller et al., 2023], statistical surveys first sample a set of municipalities and then sample participants only from these municipalities' registers [Wasmer et al., 2017, INAPP, 2022].

Our project was sparked by discussions with German sortition practitioners, who have been following a similar two-level sampling approach [Stabsstelle Bürgerräte, 2023]. Using numbers from the assembly on nutrition for illustration, they were looking for a sampling process that would (1) send out 20,000 invitation letters, (2) not send letters to more than 80 distinct municipalities at once, and (3) give each German resident an equal chance of being invited.[1]

The output of any sampling process, i.e., any probability distribution determining how many invitations to send to each municipality, can be represented in a graphical form, which we illustrate in Figure 1. To sample from this distribution, one draws a number $\rho \in [0, 1)$ uniformly at random, and considers the vertical line at this position (dashed in the figure). This line intersects the shapes in the diagram, each of which is labeled with a municipality, and the number of letters sent to a municipality is equal to the total height of the municipality's shapes at the vertical line.[2] Without loss of generality, the selection within each municipality is uniform without replacement. In this representation, the practitioners' requirements are easy to express: (1) the total height of the figure at each vertical stripe should be 20,000 letters, (2) no vertical stripe should intersect with more than 80 shapes, and (3) the total area of a municipality's shapes (i.e., its expected number of received letters) must be proportional to its population.

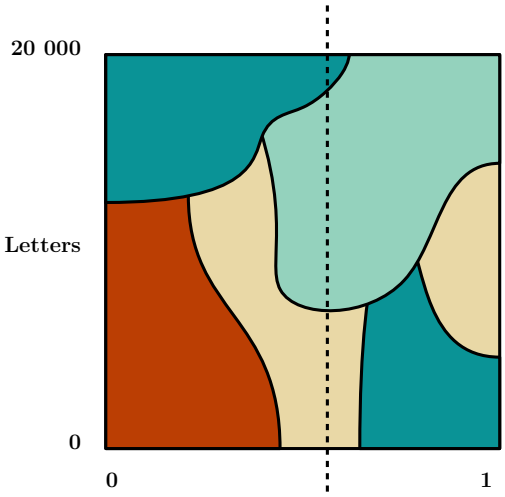

Figure 1: Graphical representation of sampling process.

A final requirement is that (4) the number of letters received by each municipality (or, the height of the municipality's shapes in any vertical strip) has an upper bound. Indeed, the municipality's population — which can be as low as 9 inhabitants in the case of Germany — is definitely an upper bound, and many municipalities are moreover reluctant to allow sampling of more than about 10% of their population due to privacy concerns. In survey sampling, such upper constraints are not present because it is possible to upweight a resident in the analysis, effectively sampling them more than once. As a result, the solution used in survey sampling — sampling municipalities with *probability proportional to size* [Brewer and Hanif, 1983], so that each vertical stripe consists of 80 equal-height layers — does not apply to assembly selection.

Whereas practitioners have so far relaxed conditions (1) and (2) [Stabsstelle Bürgerräte, 2023] due to limitations in available methods, we show that all desiderata can, in fact, be satisfied by moving beyond rectangular shapes to more flexible geometric constructions.

[1] In fact, assembly organizers break down the sampling into 42 sampling processes of this form, one for each federal state and category of municipality size. For exposition, we focus on an individual such problem, and consider the national level in Section 6.

[2] Clearly, the x-axis ordering of the diagram is arbitrary. All that we need is that each color's union of shapes is measurable.

Our Results and Techniques We begin by formulating our task as an optimization problem, MinFeasibleCities, which seeks a probability distribution satisfying the four conditions while minimizing the number of contacted municipalities. Although MinFeasibleCities is NP-hard, we provide a pseudo-polynomial time algorithm and an additive 1-approximation, both based on separation oracles for a linear programming formulation.

Since minimizing municipalities is only one of several practical goals, we introduce additional criteria. We first propose *ex-post monotonicity*, which states that, among the contacted municipalities, larger ones should receive at least as many letters as smaller ones. We present GreedyEqual, a natural algorithm that achieves ex-post monotonicity and an additive 2-approximation under mild assumptions.

Whereas GreedyEqual promotes balanced letter allocations, it is natural to strengthen monotonicity to *ex-post proportionality*, which states that a municipality's number of letters received scales with its size. We capture different proportionality goals through *target letter functions* and develop two algorithms to pursue them: an optimal method based on integer linear programming and a simpler heuristic.

Finally, we evaluate all algorithms on data from the German Citizens' Assembly on Nutrition [Stabsstelle Bürgerräte, 2023]. Since the selection is applied independently within 42 subgroups, we show how to lift the notion of target letters from the local to the global level. Our algorithms offer practical solutions that can accommodate a wide range of real-world requirements.

Related Work By contributing to the first stage of the assembly selection pipeline in practice, our work is complementary to, but technically independent from, algorithms for selecting the final assembly from those accepting the invitation. Flanigan et al. [2021a] developed an optimization-based algorithm for this task; subsequent work studied transparent ways of drawing from the algorithm's computed probability distribution [Flanigan et al., 2021b], incentives for misrepresentation [Flanigan et al., 2024, Baharav and Flanigan, 2024], accounting for self-selection bias [Flanigan et al., 2020], and the replacement of assembly members who drop out later [Assos et al., 2025].

Other works have studied sortition algorithms that directly draw the assembly from the population and resulting theoretical properties. These works study the variance of representation of features in the assembly [Benadè et al., 2019], the social welfare if assembly members participate in a sequence of binary majority votes [Meir et al., 2021], axioms and approximation bounds on the proximity of assembly members to the population in a metric space [Ebadian et al., 2022, Ebadian and Micha, 2025, Caragiannis et al., 2024], and a proposed hierarchy of interconnected assemblies [Halpern et al., 2025]. Do et al. [2021] study an online selection problem motivated by citizens' assemblies, in a random-dial methodology resembling the recruitment process in France.

2 The Theoretical Model

We are given n cities[3] and a fixed number of letters $\ell \in \mathbb{N}$ to allocate. Each city has a population $\pi_i \in \mathbb{R}$ and we assume normalization wlog, i.e., $\sum_{i\in[n]} \pi_i = 1$. We also write $\vec{\pi} = (\pi_1, \ldots, \pi_n)$. Every city has a maximum number of letters it can receive, denoted by $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$. We assume $\pi_1 \leq \cdots \leq \pi_n$, $u_1 \leq \cdots \leq u_n \leq \ell$. A letter allocation is a vector $a \in \mathbb{R}^n_{\geq 0}$ with the property that $\sum_{i\in[n]} a_i = \ell$ and $0 \leq a_i \leq u_i$ for all $i \in [n]$.[4] An allocation is *t-bounded* if at most t cities receive a non-zero number of letters; let A_t denote the set of all such allocations. Given an instance

[3]For brevity, we use 'cities' as a synonym for 'municipalities'.

[4]For $k \in \mathbb{N}$ let $[k] = \{1, \ldots, k\}$ and $[k]_0 = \{0, \ldots, k\}$.

of our problem $(\vec{\pi}, \vec{u}, t)$, FeasibleCities describes the problem of deciding whether there exists a probability distribution $\mathcal{D}$ over A_t such that

$$\mathbb{E}_{a\sim\mathcal{D}}[a_i] = \pi_i \cdot \ell \text{ for all } i \in [n]. \tag{1}$$

We also refer to a probability distribution respecting Eq. (1) as *ex-ante fair*. MinFeasibleCities describes the corresponding optimization problem of finding the minimum t such that the answer to FeasibleCities is yes.

Though letter allocations are integral in practice, i.e., $a \in \mathbb{N}^n$, this restriction is wlog for FeasibleCities since any distribution over fractional allocations for t can be turned into a distribution over t-bounded integral allocations with the same ex-ante properties, through dependent randomized rounding [Gandhi et al., 2006]. For convenience, we assume A_t to be integral in Section 3 and fractional in Section 4. We assume $\pi_i \ell \leq u_i$ for all $i \in [n]$, which is a necessary condition for the existence of an ex-ante fair distribution (for any t) due to the upper bounds (see Lemma 1). Through the paper, we refer to the following running example:

Example 1. *Distribute $\ell = 60$ letters over $n = 8$ cities. The city sizes and upper bounds are $\vec{\pi} = \frac{1}{360} \cdot (10, 10, 40, 40, 40, 50, 70, 100)$ and $\vec{u} = 180 \cdot \vec{\pi} = (5, 5, 20, 20, 20, 25, 35, 50)$.*

While Section 3 studies city sampling through the lens of the optimization problem defined above, Sections 4 and 5 motivate and define additional desirable concepts: *ex-post monotonicity*, *ex-post proportionality*, and *binary outcomes*.

3 The MinFeasibleCities Problem

In this section, we show that, though FeasibleCities is NP-hard, it is only barely a hard problem, in the sense that pseudopolynomial time computation, or a slack of a single city suffice to overcome this complexity barrier. We defer all missing proofs to Appendix A.

We start by showing a simple lower bound that will be helpful throughout the paper. To this end, we define $w_i = \frac{\pi_i \ell}{u_i}$ for all $i \in [n]$, which yields a lower bound on the *selection probability* of a city (also interpreted as the minimum *width* within our illustrations).

Lemma 1. *For any instance $(\vec{\pi}, \vec{u}, t)$, and an ex-ante fair probability distribution $\mathcal{D}$ over A_t, it holds that*

(i) $\Pr[a_i > 0] \geq w_i$ *for all* $i \in [n]$, *and*

(ii) $t \geq \sum_{i\in[n]} w_i$.

For Example 1, Lemma 1 shows that t must be at least 3 since the minimum total width of all cities is $\sum_{i\in[n]} w_i = \frac{8}{3}$.

In the appendix, we show that FeasibleCities is NP-hard via a reduction from Partition. In a nutshell, this reduction constructs an instance of our problem, in which all allocations in the support of a t-bounded, ex-ante fair distribution must give half of the cities 0 letters and half their upper bound u_i. The question whether any such allocation assigns exactly ℓ letters is exactly Partition.

Theorem 2. *FeasibleCities is NP-hard.*

Since we reduce from PARTITION, which admits a pseudo-polynomial time algorithm, it is natural to ask whether our problem does too. To show that this is the case, we formulate the problem as a linear program with one variable x_a for each integral allocation $a \in A_t$. The LP searches for a fair distribution over these, with x_a representing the probability assigned to allocation a. The first constraint ensures that the probabilities sum to at most 1; the second enforces fairness. Both hold with equality in any feasible solution, but are written as inequalities for clarity in the dual.

$$
\begin{aligned}
\textbf{Primal:} \quad & \text{minimize } 0 \\
& \text{subject to } \sum_{a \in A_t} x_a \leq 1, \\
& \qquad \sum_{a \in A_t} x_a a_i \geq \pi_i \ell \qquad \text{for } i \in [n], \\
& \qquad x_a \geq 0 \qquad \text{for } a \in A_t. \\
\textbf{Dual:} \quad & \text{maximize} \sum_{i \in [n]} \pi_i \ell y_i - y \\
& \text{subject to } \sum_{i \in [n]} a_i y_i \leq y \qquad \text{for } a \in A_t, \\
& \qquad y, y_i \geq 0 \qquad \text{for } i \in [n].
\end{aligned} \tag{2}
$$

We aim to decide whether the primal LP is feasible, which is the case iff the dual LP admits no solution with positive objective value (which could be scaled to show that the dual value is unbounded). We add a constraint to the dual requiring a strictly positive objective value.

Though the resulting system has exponentially many constraints, its feasibility can be decided with the ellipsoid method [Grötschel et al., 1993] provided we can implement a *separation oracle* for the dual: given a vector $\big((y_i)_{i \in [n]}, y\big)$, we must decide whether it is feasible for the modified dual or return a violated constraint. We show that this separation problem can be solved in pseudo-polynomial time using a knapsack-style dynamic program.

Theorem 3. *There exists a pseudo-polynomial time algorithm for* FEASIBLECITIES.

More surprisingly, we can construct a polynomial-time *approximate separation oracle*, in the following, strong sense: given a vector $\big((y_i)_{i \in [n]}, y\big)$, our oracle either determines that the vector satisfies all dual constraints or identify a violated constraint of type (2), but for for some allocation $a \in A_{t+1} \supseteq A_t$ rather than in A_t. As Schulz and Uhan [2013] show, the ellipsoid method with such an approximate oracle can determine either that the dual above is unbounded (so the primal is infeasible) or that the dual for $t+1$ cities is bounded (hence, the primal for $t+1$ cities is feasible). By applying this algorithm to increasing values of t until a feasible primal is found, we can find the lowest possible number of contacted cities, up to perhaps one additional city.

Theorem 4. *There exists a polynomial-time algorithm that is an additive* 1*-approximation to* MIN-FEASIBLECITIES.

While the above algorithms are theoretically tractable, the ellipsoid method is a famously impractical algorithm.[5] Moreover, these algorithm may yield highly unintuitive allocations that would

[5] Although lacking theoretical guarantees, combining our (or similar) separation oracles with the simplex method can still lead to practical algorithms (see COLUMNGENERATION in Section 5).

be difficult to justify in practice. For example, a large city might receive significantly fewer letters than a smaller one ex post, or only small cities might be selected while all larger ones are excluded. We now shift our focus from mere feasibility to fair distributions that uphold additional desirable properties, all while keeping t low.

4 A Simple and Monotone Approximation

In this section, we aim for *ex-post monotonicity*. A fractional allocation a is called *monotone* if $a_i \geq a_j$ whenever $i > j$. A probability distribution is ex-post monotone if its support consists of monotone allocations. We present a simple 2-approximation for MinFeasibleCities that yields ex-post monotone distributions under mild assumptions. The algorithm is inspired by πps sampling [Brewer and Hanif, 1983]: given an instance $(\vec{\pi}, \vec{u}, t)$, one samples t cities with probabilities proportional to $\vec{\pi}$ without replacement and assigns ℓ/t letters to each. While this would violate cities' upper bounds, our algorithm can be viewed as a minimal adjustment to πps sampling to ensure feasibility.

Our algorithm, GreedyEqual, is best understood through its geometric interpretation. The algorithm processes cities in increasing order of size and starts by attempting to place a $\pi_i\ell$-area rectangle of height ℓ/t. If this violates the city's upper bound, it instead places a rectangle of height u_i. It then proceeds to place the next rectangle to the right. Once the first layer is filled, GreedyEqual moves to the next layer, now aiming to keep the height of rectangles at the remaining vertical space divided by $t-1$. This ensures that later (and thus larger) cities can receive at least as many letters as those already placed. We remark that, starting from the second layer, cities may receive a set of rectangles summing to $\pi_i\ell$ instead of a single rectangle, which is due to shifts in lower layers. See Figure 2a for an illustration.

To formalize GreedyEqual, we introduce a second type of illustration, which is a flattened version of the illustration in Figure 2a. This illustration is formalized by functions λ_i for each $i \in [n]$ that are defined on the interval $[0, t)$. The value $\lambda_i(x)$ corresponds to the height of the rectangle that the algorithm draws for city i in layer $\lfloor x \rfloor$ (0-indexed) and at position $x - \lfloor x \rfloor$ of the stacked picture. (Note that for any position $x \in [0, t)$ this value will be non-zero for exactly one city as the algorithm draws for one city at a time.) We illustrate these functions in Figure 2b.

When the algorithm draws at position x in the flat picture, it needs to know the height of all rectangles that were placed at some value $y \leq x - 1$ with $y \equiv x \pmod 1$. We define

$$\Lambda(x) = \sum_{y \leq x,\, y \equiv x \pmod 1} \sum_{j \leq i} \lambda_j(y).$$

Last, we define $\mu_i(x)$, describing the height of the rectangle to be drawn, given that we place city i at position x,

$$\mu_i(x) = \begin{cases} \min\left(u_i, \frac{\ell - \Lambda(x-1)}{t - \lfloor x \rfloor}\right) & \text{for } x \in [0, t) \\ u_i & \text{for } x \geq t, \end{cases}$$

and are now ready to formalize GreedyEqual:

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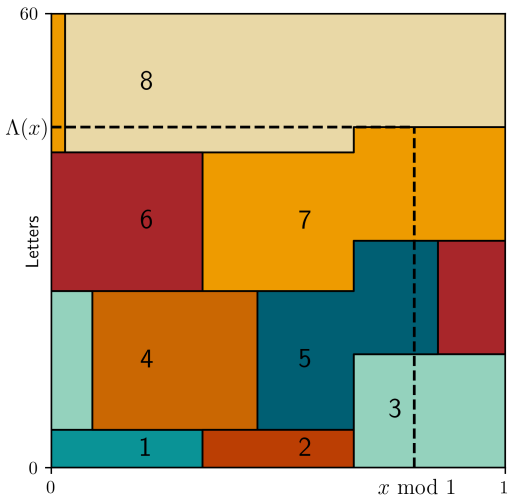
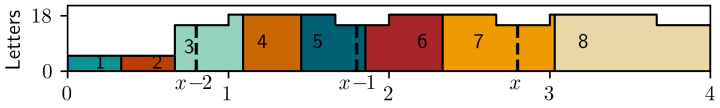
procedure GreedyEqual($\vec{\pi}, \vec{u}, t$)
  $x \leftarrow 0, i \leftarrow 1$
  while $i \leq n$ **do**
    let $y \geq x$ such that $\int_x^y \mu_i(z)dz = \pi_i\ell$
    $\lambda_i(z) \leftarrow \mu_i(z)$ for $z \in [x, y)$, $x \leftarrow y$, $i \leftarrow i + 1$
  if $x = t$ **then return** $(\lambda_i)_{i \in [n]}$ **else** "fail"



(a) Illustration GreedyEqual for Example 1 for $t = 4$.



(b) Illustration of the functions λ_i for all $i \in [n]$, where each λ_i is indicated by a different color.

Figure 2: GreedyEqual applied to Example 1.

GreedyEqual can fail in two ways. First, it may terminate with $x > t$, meaning it requires more than t layers and thus does not yield a t-bounded distribution. In Theorem 8, we bound the optimum of MinFeasibleCities in this case. Second, and more subtly, the area assigned to a city may be so wide that it overlaps across layers, leading to an allocation that exceeds the upper bound of the city. This can only happen when a city is *oversized*, i.e., when $\pi_i > 1/t$. While such cities appear in parts of our data, they always have upper bounds well above ℓ, making this a non-issue in practice. We formalize the following assumption:

Assumption 1. *For any oversized city i, $u_i \geq \ell$.*

In our dataset, Assumption 1 is satisfied as long as $t \leq 420$, far above the past choice of $t = 80$.

Theorem 5. *Under Assumption 1, GreedyEqual always returns an ex-ante fair and t-bounded probability distribution (if it succeeds).*

In instances without oversized cities we furthermore guarantee monotonicity. In our data we do not observe any monotonicity violation, even for oversized cities.

Theorem 6. *For instances without oversized cities, GreedyEqual is ex-post monotone.*

[5]For ex-post monotonicity, the relaxation to fractional allocations is not quite wlog, but any fractional monotone allocation can be decomposed into a distribution over integral allocations that are monotone up to one letter.

Before proving the additive 2-approximation, we provide insight into the structure of GreedyEqual's solutions. We say that GreedyEqual *selects the average* at position $x \in [0,t)$ if $\lambda_i(x) = \frac{\ell - \Lambda(x-1)}{t - \lfloor x \rfloor}$ (rather than $\lambda_i(x) = u_i$), where i is the unique city with $\lambda_i(x) > 0$.

Lemma 7. *Independent of whether GreedyEqual succeeds, the following holds:*

(i) If GreedyEqual selects the average at $x \in [0,t)$, then it selects the average at all $y \in [x, \lceil x \rceil)$ as well as all $y \in [x,t)$ with $y \equiv x \pmod 1$.

(ii) The function $\Lambda(x)$ is non-decreasing on $[0,t)$.

Theorem 8. *Under Assumption 1, GreedyEqual is an additive 2-approximation for MinFeasibleCities.*

Proof (first part). Let $(\vec{\pi}, \vec{u}, t)$ be an instance of our problem such that GreedyEqual fails. We show that $\sum_{i \in [n]} w_i > t - 2$, which by Lemma 1 (ii), implies that the optimal budget for MinFeasibleCities is at least $t - 1$.

To gain intuition for the proof, consider the following thought experiment: imagine scaling each city's shapes so that it maintains its total area but reaches its maximum height u_i, attaining its minimum width w_i. How much width do we lose in total? The original sum of widths exceeds t; we show that even after scaling, the total width remains strictly greater than $t - 2$. While it may seem natural to scale each city individually, our analysis instead partitions the stacked picture into "columns" and scales each column separately.

Let $\mathcal{I}$ be a partition of the interval $[0,1)$ with the property that all functions λ_i are constant along each interval $I \in \mathcal{I}$. Now, consider the interval in $\mathcal{I}$ that starts at 0. For all $k \in [t-1]_0$ let $j(k)$ be the unique city with $\lambda_{j(k)}(k) > 0$. From now, we drop the position and write $\lambda_{j(k)}$ instead of $\lambda_{j(k)}(k)$. Since GreedyEqual failed, we know that $\Lambda(x) < \ell$ for some $x \in [t-1, t)$. Moreover, by the monotonicity of Λ (Lemma 7 (ii)) we know that $\Lambda(t-1) < \ell$. By Lemma 7 (i) it holds that $\lambda_{j(k)} = u_{j(k)}$ for all $k \in [t-1]_0$.

Now consider an arbitrary interval $[\alpha, \beta) \in \mathcal{I}$. For each $k \in [t-1]_0$, let $i(k)$ be the unique city with $\lambda_{i(k)}(k+\alpha) > 0$. We write $\lambda_{i(k)}$ for $\lambda_{i(k)}(k+\alpha)$. See Figure 3 for an illustration. The original total width in column $[\alpha, \beta)$ is $(\beta - \alpha)t$. We show that scaling each subarea to its maximum height yields a total width greater than $(\beta - \alpha)(t-2)$. Since the factor $(\beta - \alpha)$ is irrelevant to our argument, we drop it.

Claim. *It holds that $\sum_{k=0}^{t-2} \frac{\lambda_{i(k)}}{u_{i(k)}} > t - 2$.*

Proof of the claim. Since GreedyEqual processes cities with increasing indices, it holds that $i(k) \le j(k+1)$ for all $k \in [t-2]_0$. Thus:

$$\lambda_{i(k)} \le u_{i(k)} \le u_{j(k+1)} = \lambda_{j(k+1)}. \tag{3}$$

Let t' be the first index for which $\lambda_{i(t')} = \frac{\ell - \Lambda(t'-1+\alpha)}{t - t'}$. If no such index exists, then we know that $\lambda_{i(k)} = u_{i(k)}$ for all $k \in [t-2]_0$ and the claim follows trivially. In the example in Figure 3 it holds that $t' = 3$. We define $\ell_i = \sum_{k=t'}^{t-1} \lambda_{i(k)}$ and $\ell_j = \sum_{k=t'+1}^{t-1} u_{j(k)}$. Note that $\ell_i > \ell_j$, since $\Lambda(t' - 1 + \alpha) < \Lambda(t')$. By Lemma 7 (i):

$$\lambda_{i(k)} = \frac{\ell_i}{t - t'} \text{ for all } k \in \{t', \dots, t-1\} \tag{4}$$

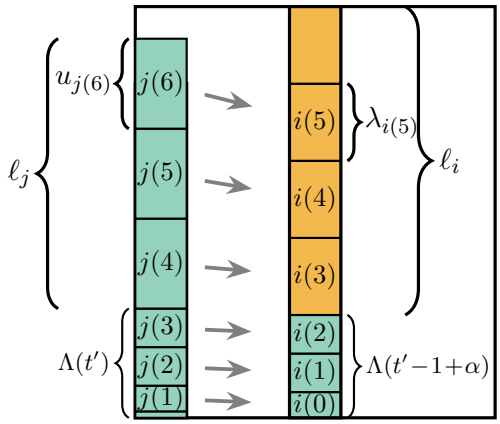


Figure 3: Situation in the proof of Theorem 8. In teal areas, cities receive their upper bounds u_i and in orange areas they receive less than u_i. An arc indicates $u_{j(k)} \geq u_{i(k-1)}$.

We are now ready to prove the claim

$$\begin{aligned}
\sum_{k=0}^{t-2} \frac{\lambda_{i(k)}}{u_{i(k)}} &= t' + \sum_{k=t'}^{t-2} \frac{\lambda_{i(k)}}{u_{i(k)}} \\
&\overset{(4)}{=} t' + \frac{\ell_i}{(t-t')} \sum_{k=t'}^{t-2} \frac{1}{u_{i(k)}} \\
&\overset{(\star)}{\geq} t' + \frac{\ell_i}{(t-t')} \frac{(t-t'-1)^2}{\sum_{k=t'}^{t-2} u_{i(k)}} \\
&\overset{(3)}{\geq} t' + \frac{\ell_i}{(t-t')} \frac{(t-t'-1)^2}{\sum_{k=t'+1}^{t-1} u_{j(k)}} \\
&> t' + \frac{\ell_j}{t-t'} \frac{(t-t'-1)^2}{\ell_j} \\
&= t' + t - t' - 2 + \frac{1}{t-t'} > t-2,
\end{aligned}$$

where $(\star)$ follows from the fact that the arithmetic mean is at least the harmonic mean (applied to the values $\frac{1}{u_i}$). ■

It remains to apply the above claim to all columns and conclude $\sum_{i\in[n]} w_i > t-2$. We refer to the appendix. □

We also show that our upper bound for the approximation guarantee of GreedyEqual is tight:

Theorem 9. *Even under Assumption 1, GreedyEqual is not an additive* 1*-approximation for MinFeasibleCities.*

5 Ex-post Proportionality

Ex-post monotonicity ensures that after randomization, larger selected cities receive at least as many letters as smaller ones. However, it does not guarantee that larger cities receive *more* letters. For

example, a city with millions of inhabitants might still receive the same number of letters as one with only tens of thousands — behavior that GreedyEqual in fact encourages. We explore how both the selection probability and the number of letters a city receives (if selected) can grow with the population. To achieve this, we introduce the more general concept of *target letters*.

We assign each city i a number τ_i of target letters, that it should receive if it is selected. This, in term also implies a target selection probability $\frac{\pi_i \ell}{\tau_i}$ for that city. If we let the target letters grow proportionally to the population, then each city gets selected with the same probability. If, on the other hand, the target is equal for all cities, then the target selection probability grows proportionally to the population, which is close to what GreedyEqual achieves. It seems natural to allow for target functions in between those two extremes.

We define a *target letter function* to be a monotone function f taking as input a population size π_i and outputting a target in $\mathbb{R}_{\geq 0}$. However, blindly setting targets without knowing the budget t can lead to infeasibility: For example, small targets will clearly be missed if t is very small. To mitigate this issue, we introduce a scaling factor κ and define for each city i the scaled target letters as

$$\tau_i^\kappa = \max\left(\pi_i \ell, \min\left(u_i, \kappa f\left(\pi_i\right)\right)\right), \tag{5}$$

which makes sure that target letters do not exceed u_i and the target selection probability does not exceed 1. We then determine the value κ such that the total target selection probability (or width) satisfies $\sum_{i\in[n]} \frac{\pi_i \ell}{\tau_i^\kappa} = t$ and set $\tau_i = \tau_i^\kappa$ for all $i \in [n]$. A total width of at most t is a necessary condition for the targets to be achievable but is far from sufficient due to the more complex structure of the problem.

We suggest $f(x) = \sqrt{x}$ as a particularly natural target letter function since it allows target letters and target selection probability to scale in equal measure. We introduce two methods that take as input an instance and the target letters and aim to construct a fair distribution meeting the targets. As in Section 4, we allow for distributions over fractional allocations for the sake of simplicity, which immediately approximates integral ex-post proportionality up to one letter.

Column generation Recall our linear programming approach from Section 3, which we used for deciding whether an instance is feasible or not. It is natural to add an objective function to this LP to minimize, in expectation, a measure of deviation from the targets. Specifically, we minimize $\sum_{a\in A_t} x_a \varphi(a)$, where φ measures the total relative deviation from the targets, i.e.,

$$\varphi(a) = \sum_{i\in[n], a_i>0} \frac{|\tau_i - a_i|}{\tau_i}.$$

This objective penalizes the same absolute deviation from the target more heavily for smaller cities than for larger ones. To optimize the resulting primal LP, we again design a separation oracle for the dual LP (a process also termed *column generation*). This time, the separation problem is more complex, and we formulate a mixed integer linear program to solve it (Appendix B). Though not polynomial-time, state-of-the-art solvers scale to large problems in practice.

While ColumnGeneration is optimal with respect to the target letters, the resulting distributions have little visual structure (e.g., see Figure 4c), and the algorithm's reliance on optimization solvers makes them hard to explain to the public. We introduce an alternative approach that is arguably more transparent, while still aiming to meet the target letters.

Bucket Approach The idea of Buckets is to partition the cities into t disjoint sets (the *buckets*), such that we can then sample exactly one city from each bucket. Each bucket has a *height*, which

determines how many letters the selected city from that bucket receives. Within each bucket, we thus need to sample proportional to size. By ex-ante fairness, the height of a bucket $B \subseteq [n]$ is determined by its elements $h = \sum_{i \in B} \pi_i \ell$. To approximate the target letters $\vec{\tau}$, we define the buckets such that the target letters of each city are close to the height of the bucket it belongs to.

To achieve this, we fill the buckets iteratively with cities in increasing order of their size. We move on to the next bucket if adding another city would either (i) increase the total target probability of all cities in the bucket above one, or (ii) would increase the height of the bucket above the maximum number of letters of its smallest city. See Appendix B.

The bucket approach has the advantage of producing easily explainable distributions (see, e.g., Figure 4b). In particular, it satisfies the *binary outcome* property: each city knows in advance how many letters it will receive if selected. While the method does not guarantee ex-post monotonicity in the worst case, we observe no violations in our data. Moreover, it ensures that selected cities are distributed somewhat evenly across cities of different sizes. On the downside, the approach lacks worst-case approximation guarantees.

Theorem 10. *For any targets and constant c, Buckets is not an additive c-approximation for MinFeasibleCities.*

6 Towards Practice

We aim to apply one of our algorithms in future implementations of citizens' assemblies, particularly in Germany. To this end, we tested them on the data used for the assembly on nutrition [Stabsstelle Bürgerräte, 2023], where $\ell = 20\text{K}$ letters were sent, the outreach budget was $t = 80$, and there are $n = 100\,755$ cities with a total population of 84 M. Following suggestions from practitioners, we define the maximum number of letters a city can receive as follows: 50% of the population for cities under 500 inhabitants, 10% for those over 2 500, and 250 for populations in between.

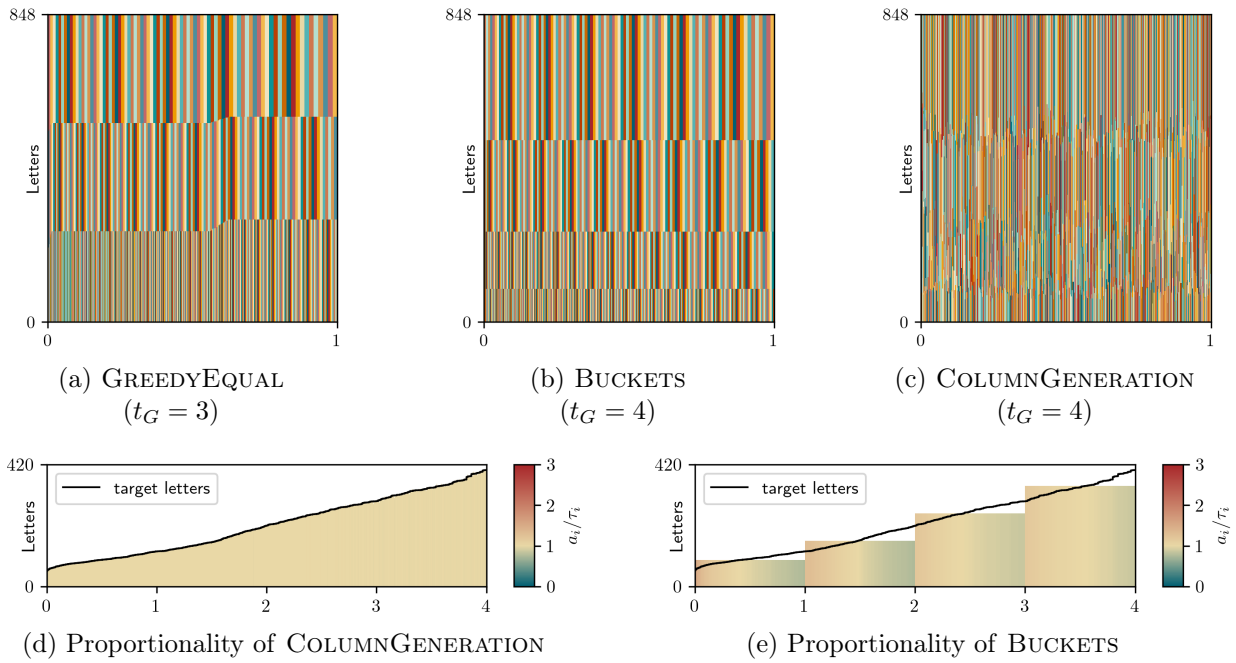
In this recent assembly, practitioners divided the country into 42 groups, based on the 16 federal states and on three city size classes ($[0, 20\text{K})$, $[20\text{K}, 100\text{K})$, $[100\text{K}, \infty)$),[6] and sampled the letter allocation per group. This stratification ensures sufficient numbers of invitations within each group for forming the assembly in the second stage of selection. Since the same grouping will likely be used for future assemblies, we test our algorithms in this setup.

Apportionment via Global Targets While a group's number of letters is just proportional to its population, we must decide how to allocate the outreach budget $t = 80$ across groups. Let $\mathcal{G}$ be the partition of $[n]$ into groups. Blindly apportioning the outreach budget t into group budgets t_G for each $G \in \mathcal{G}$ and then applying our algorithms is not ideal for meeting letter targets: Similarly sized cities in different groups may receive vastly different numbers of letters when selected, as this number depends on t_G.

We introduce the concept of *global targets*, which help finding an apportionment that keeps letter targets comparable across groups. Given a target letter function (for ColumnGeneration and Buckets we use $f(x) = \sqrt{x}$ and for GreedyEqual we use a constant function), we compute the global target letters τ_i by finding a scaling factor κ such that the corresponding target widths $\omega_i = \frac{\pi_i \ell}{\tau_i^\kappa}$ sum up to $t = 80$ (compare Section 5).

However, as argued in Section 5, within each group G, we need to rescale $\sum_{i \in G} \omega_i$ to a width of t_G to obtain sensible *local targets*. To keep the amount of rescaling required low (and, in turn, local targets close to global targets), we want to assign each group an integer budget t_G close to their

[6] Some states consist of only a single or two large cities.



(a) GreedyEqual ($t_G = 3$)

(b) Buckets ($t_G = 4$)

(c) ColumnGeneration ($t_G = 4$)

(d) Proportionality of ColumnGeneration

(e) Proportionality of Buckets

Figure 4: Probability distributions for the group of small cities in the state of Niedersachsen.

fractional target width $\sum_{i \in G} \omega_i$. This is an apportionment problem, for which we use an adjustment of Adam's apportionment method [Balinski and Young, 2001]. For details, see Appendix C.

Meeting Local Targets After finding the apportionment as described above, we test our algorithms (GreedyEqual, ColumnGeneration, and Buckets) on these 42 groups. All algorithms find distributions for the apportioned t_G, and run in a practical amount of time on consumer hardware. This shows that our algorithms scale to practical problems and are plausible contenders for deployment. We defer results and detailed discussions to Appendix D and display the distributions for one group in Figures 4a to 4c.

Figures 4d and 4e visualize how well ColumnGeneration and Buckets meet their local targets. The figure is the result of ordering all rectangles from Figures 4b and 4c by the city they represent and lining them all up next to each other in increasing order of city sizes. Each rectangle's color represents how close its height is to the target letters of that city. We plot the local targets of cities in black. In this instance, ColumnGeneration meets the target letters almost perfectly, which is true for most of the groups (more precisely, 35 out of 42 and in particular for all groups with $t_G \geq 3$). Buckets approximates the target letters, with the smaller cities within each bucket receiving slightly too many, and the larger ones slightly too few letters. Buckets struggles when there are very small cities, since the smallest city in a bucket bounds its height and limits the number of letters to the other cities in the bucket. This effect appears in 5 out of the 42 groups. Both approaches align more closely with local targets for higher values of t_G.

Meeting Global Targets We observe that the local targets of groups with $t_G > 1$ never deviate from the global targets by more than a factor of 1.5. For groups with $t_G = 1$ the local targets are independent of the target function, as every city must receive all letters of this group when selected, which can lead to arbitrarily high deviations from the global targets. In particular, many of the medium- and large-size groups have a low total share of the population, which leads to a

target width significantly below 1. Since each group must have $t_G \geq 1$, these groups are assigned $t_G = 1$, resulting in local targets letters that can be much lower than the global targets. We present a visualization of global and local targets in Figures 51 to 54 in the appendix.

Germany holds assemblies nationally and at the state level. Figure 55 (Appendix D) shows results for Baden-Württemberg, a particularly active state.

7 Discussion

We introduced a novel two-stage sampling problem, motivated by the practical demands of selecting citizens' assemblies. Our results offer a solid algorithmic foundation and give rise to two compelling open questions: Does there exist an ex-post monotone additive 1-approximation algorithm? And can the representation of city groups — currently addressed via partitioning — be integrated more directly into the model? GreedyEqual and Buckets already ensure the ex-post representation of cities of different sizes by design, and one might envision a two-dimensional sampling framework, as is often used in survey sampling [Cox, 1987].

While these questions offer exciting directions for theory, our focus remains on practical impact. As Germany's newly elected government just reaffirmed its commitment to citizens' assemblies [CDU et al., 2025], our work offers a suite of implemented algorithms, striking distinct, favorable tradeoffs between different practical desiderata. Based on our discussions with practitioners, we are optimistic that they can soon be used to sample real assemblies.

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A Missing Proofs

Lemma 1. *For any instance $(\vec{\pi}, \vec{u}, t)$, and an ex-ante fair probability distribution $\mathcal{D}$ over A_t, it holds that*

(i) $\Pr[a_i > 0] \geq w_i$ *for all* $i \in [n]$*, and*

(ii) $t \geq \sum_{i \in [n]} w_i$.

Proof. Let $\mathcal{D}$ be an ex-ante fair t-bounded probability distribution. To prove Property (i), rewrite ex-ante fairness as

$$\pi_i \ell = \mathbb{E}[a_i] \leq \Pr[a_i > 0] \cdot u_i,$$

which is equivalent to

$$\Pr[a_i > 0] \geq w_i = \frac{\pi_i \ell}{u_i}. \tag{6}$$

To show Property (ii), we rewrite the sum of selection probabilities as

$$t \geq \sum_{a \in A_t} \Pr[a] \sum_{i \in [n]} \mathbb{1}[a_i > 0] = \sum_{i \in [n]} \Pr[a_i > 0] \geq \sum_{i \in [n]} w_i,$$

which proves the lemma statement. □

Theorem 2. *FeasibleCities is NP-hard.*

Proof. We reduce from the NP-hard problem EqualCardinalityPartition [Garey and Johnson, 1979], where we are given $2k$ elements with positive integer weights $x_1, \ldots, x_{2k}$ and the task is to decide whether there exists a subset $S \subseteq [2k]$ with the property that

$$\sum_{i \in S} x_i = \frac{1}{2} \sum_{i \in [2k]} x_i.$$

We create an instance of FeasibleCities by introducing $n = 2k$ cities with populations $\pi_i = \frac{x_i}{\sum_{j \in [n]} x_j}$ and upper bounds $u_i = x_i$ for all $i \in [n]$. Moreover, we set $\ell = \frac{1}{2} \sum_{j \in [n]} x_j$ and $t = k$. We now prove the equivalence of the reduction.

First, assume that the partition instance is a yes-instance. That is, there exists a subset $S \subseteq [2k]$ such that S and $[2k] \setminus S$ have equal weight and are each of cardinality k. Now, consider the probability distribution $\mathcal{D}$ that, with probability $\frac{1}{2}$ selects the allocation $a_i = u_i$ for all $i \in S$ (and $a_i = 0$ for all $i \in [n] \setminus S$), and with probability $\frac{1}{2}$ selects the allocation $a_i = u_i$ for all $i \in [n] \setminus S$ (and $a_i = 0$ for all $i \in S$). This allocation is ex-ante fair and t-bounded. Hence, our FeasibleCities instance is a yes-instance.

For the other direction, let our FeasibleCities instance be a yes-instance. We then derive two properties that have to hold for any ex-ante fair and t-bounded probability distribution. First, note that by construction $w_i = \frac{1}{2}$ for all $i \in [n]$. Moreover, as we have argued in Lemma 1, it holds that

$$t \geq \sum_{i \in [n]} \Pr[a_i > 0] \geq \sum_{i \in [n]} w_i = \frac{n}{2} = t,$$

and therefore $\Pr[a_i > 0] = w_i = \frac{1}{2}$ for all cities. In particular, this also implies that for any allocation in the support of $\mathcal{D}$ there exist exactly t cities with $a_i = u_i$ while for all others it holds that $a_i = 0$. Thus, consider any allocation in the support of $\mathcal{D}$ and let S be the set of cities with non-zero letters. Since S is of weight $\sum_{i \in S} u_i = \ell = \frac{1}{2}$ and of cardinality k, this proves the existence of an equal-cardinality partition. □

Theorem 3. *There exists a pseudo-polynomial time algorithm for* FeasibleCities.

Proof. Our goal is to decide whether the primal LP is feasible. Since the dual is always feasible (set $y_i = y = 0$ for all $i \in [n]$), this is equivalent to deciding whether the dual LP is bounded. Note that the dual is unbounded if and only if it has any solution with positive objective value (we can always scale the y_i's and y). Thus, we can add the constraint

$$\sum_{i \in [n]} \pi_i \ell y_i - y = 1$$

and ask whether the dual LP is non-empty. For deciding non-emptiness of an LP it suffices to have access to a separation oracle in order to employ the Ellipsoid method [Grötschel et al., 1993].

The dual separation problem is the following: Given a dual solution $((y_i)_{i \in [n]}, y)$, decide whether there exist a dual constraint that is violated:

$$\sum_{i \in [n]} a_i y_i \leq y \quad a \in A_t.$$

Stated differently, is $\max_{a \in A_t} \sum_{i \in [n]} a_i y_i$ larger than y? This problem is reminiscent of a knapsack problem and can be solved with help of a dynamic program. First note that it is without loss of generality that an optimal solution a to this maximization problem gives $a_i < u_i$ to at most one city $i \in [n]$, namely to the one (among the $\leq t$ selected ones) with smallest weight y_i.

We start our algorithm by guessing the city with smallest weight that will be included the support of a, let's call this city i^*. Now, we relabel the cities such that $y_1 \geq \cdots \geq y_{i^*} > y_{i^*} \geq \ldots y_n$. For deciding which other cities will be included with their upper bound in a (which will be up to $t-1$), we now write an dynamic program, which is essentially the same as for a cardinality-constrained knapsack problem. We give it for the sake of completeness. We write the following dynamic program, where $\mathrm{DP}(0 \leq j < i^*, 0 \leq k \leq t-1, 0 \leq z \leq \ell)$ corresponds to the maximum value that we can create if we select k cities from the set $\{1, \ldots, j\}$ and allocate exactly z letter to them. Then, the recursive formulas of the dynamic program can be described as follows:

$$\mathrm{DP}(0, \cdot, \geq 0) = \mathrm{DP}(\cdot, 0, \geq 0) = \mathrm{DP}(\cdot, \cdot, 0) = 0, \mathrm{DP}(\cdot, \cdot, < 0) = -\infty$$

and

$$\mathrm{DP}(j+1, k, z) = \max\left(\mathrm{DP}(j, k-1, z - u_{j+1}) + y_i u_{j+1}, \mathrm{DP}(j, k, z)\right).$$

After filling the table, we choose the entry that maximizes $\mathrm{DP}(i^* - 1, k, z)$ among all $k \in [t-1], z \in [\ell - u_{i^*}, \ell)$, let's call this value γ_{i^*} and remember the corresponding z as z_{i^*}. We then choose i^* as to maximize $\gamma_{i^*} + (\ell - z_{i^*}) \cdot y_{i^*}$. This leads to an algorithm with running time $\mathcal{O}(n^2 t \ell)$. □

Theorem 4. *There exists a polynomial-time algorithm that is an additive* 1*-approximation to* Min-FeasibleCities.

Proof. We start this proof very similarly to the one of Theorem 3, namely, we add the constraint

$$\sum_{i \in [n]} \pi_i \ell y_i - y = 1$$

and ask whether the dual LP is non-empty. This time, we show the existence of an approximate separation oracle: Given a dual solution $((y_i)_{i \in [n]}, y)$, decide that the solution is feasible or provide one constraint of the following that is violated:

$$\sum_{i \in [n]} a_i y_i \leq y \quad a \in A_{t+1}.$$

Note that these correspond to the dual constraint for the case of t, since we replaced A_t by A_{t+1}. Schulz and Uhan [2013] prove in the appendix of their paper (Theorem A.6) that if an approximate separation oracle in combination with the Ellipsoid method yields an approximate algorithm for the deciding the emptiness of a polytope. More precisely, in our case this implies that if there exists a polynomial time algorithm for our approximate separation oracle, then the Ellipsoid method yields an algorithm that either decides that our dual LP for t is non-empty or our dual LP for $t+1$ is empty. In the former case, this implies that the primal LP for t is infeasible, in the latter case the primal LP for $t+1$ is feasible. Thus, this algorithm is an additive 1-approximation for MinFeasibleCities.

We now provide the polynomial-time algorithm for the approximate separation oracle. For a given dual solution $((y_i)_{i\in[n]}, y)$ we should either decide that

$$m_t := \max_{a\in A_t} \sum_{i\in[n]} a_i y_i \leq y \tag{7}$$

or prove that

$$m_{t+1} := \max_{a\in A_{t+1}} \sum_{i\in[n]} a_i y_i > y. \tag{8}$$

Note that $m_{t+1} \geq m_t$ holds simply because $A_t \subseteq A_{t+1}$. We can write down the following LP, which serves as a LP relaxation of the former optimization problem:

$$\begin{aligned}
\text{maximize} \quad & \sum_{i\in[n]} z_i y_i u_i \\
\text{subject to} \quad & \sum_{i\in[n]} z_i u_i = \ell, \\
& \sum_{i\in[n]} z_i \leq t \\
& 0 \leq z_i \leq 1 \quad \text{for } i \in [n].
\end{aligned}$$

The idea is that z_i captures the number of letters allocated to city $i \in [n]$, namely, $a_i = z_i u_i$. Note that in particular, if a is an optimal solution to Equation (7), then $z_i = \frac{a_i}{u_i}$ is a feasible solution to the above LP. Hence, m_t is smaller or equal than the optimal value of the LP, which we call m^*. Now, let z^* be some optimal solution to the LP that also corresponds to a basis. Since n constraints must be tight in such a solution, we know that at least $n-2$ of the constraints $0 \leq z_i \leq 1$ must be tight on one of the two sides. Thus, we have at most two variables z_i with fractional values, thus there are at most $t+1$ variables with non-zero entry. We construct a solution $a_i^* = z_i^* u_i$ for all $i \in [n]$. This solution is almost an element of A_{t+1} with the subtlety that there might be two cities that may receive a non-integral amount of letters (those with fractional z_i). Let's call these cities $i(1)$ and $i(2)$ and assume wlog that $y_{i(1)} \geq y_{i(2)}$. Then, we round $a^*_{i(1)}$ up and $a^*_{i(2)}$ down. While the corresponding z solution would not be feasible for the LP (it would violate the t-bound), a^* remains $(t+1)$-bounded and only increases the objective value of a^* wrt the y_i weights. Hence, we can assume wlog that $a^* \in A_{t+1}$.

If $m^* \leq y$, then we report that the point $((y_i)_{i\in[n]}, y)$ is feasible for our dual LP for t, which is true since $m_t \leq m^* \leq y$. On the other hand, if $m^* > y$, then we return the violated constraint from the dual constraint for $t+1$ that corresponds to $a^* \in A_{t+1}$, which is true since

$$\sum_{i\in[n]} a_i^* y_i = \sum_{i\in[n]} z_i^* u_i y_i = m^* > y,$$

which implies $m_{t+1} \geq m^* > y$. □

Theorem 5. *Under Assumption 1, GreedyEqual always returns an ex-ante fair and t-bounded probability distribution (if it succeeds).*

Proof. Note that, when GreedyEqual succeeds in particular it needs to hold that $\Lambda(x) = \ell$ for all $x \in [t-1, t)$ since both the area of the rectangle and the sum of the areas of the cities sums to ℓ.

We define the corresponding probability distribution $\mathcal{D}$ as follows: Sample $\alpha \in [0, 1)$ uniformly at random and for any $k \in [t-1]_0$, let $i(k)$ be the unique city with $\lambda_i(k+\alpha) > 0$. Then, return $a_{i(k)} = \sum_{k' \in [t-1]_0} \lambda_{i(k)}(k'+\alpha)$ for all $k \in [t-1]_0$ and $a_j = 0$ for all other cities j. Note that typically, the sum in the definition of $a_{i(k)}$ only contains one non-zero entry with the exceptions of cities that "wrap" around several layers.

We now prove that $\mathcal{D}$ is a probability distribution over A_t. The fact that $\sum_{i \in [n]} a_i = \ell$ follows since $\Lambda(t-1+\alpha) = \ell$. Moreover, a is t-bounded by construction. Lastly, we claim that $a_i \leq u_i$ for all $i \in [n]$. Here, we only have to be concerned about cities that "wrap around", since otherwise the constraint follows directly by the definition of the algorithm. Note that for any λ_i at any position, $\min\{u_i, \frac{\ell}{t}\}$ is a global lower bound. This is because the average value that is compare to u_i in the function μ_i starts out to be $\frac{\ell}{t}$ in the first round and then only grows over time. Thus, assume that $i \in [n]$ wraps around. Then, either $\frac{\pi_i \ell}{u_i} > 1$ or $\frac{\pi_i \ell t}{\ell} > 1$. The first constraint is a direct contradiction to our assumption from Section 2 since this would immediately imply that our instance is infeasible for all t. The second constraint implies that i is oversized. In this case, Assumption 1 implies that $u_i = \ell$ and therefore $a_i \leq \ell$ holds trivially.

Lastly, $\mathcal{D}$ is ex-ante fair since by construction the area in the picture for each city $i \in [n]$ is $\pi\ell$. □

Theorem 6. *For instances without oversized cities, GreedyEqual is ex-post monotone.*

Proof. Consider any interval $[\alpha, \beta)$ such that all $\lambda_i(x)$ are constant for all $x \in [\alpha, \beta)$. For $k \in [t-1]_0$ let $i(k)$ be the unique city with $\lambda_{i(k)}(k+\alpha) > 0$. We show that

$$\lambda_{i(0)} \leq \lambda_{i(1)} \leq \cdots \leq \lambda_{i(t-1)}.$$

Let $t' \in [t'-1]_0$ be the largest index such that $\lambda_{i(t')} = u_{i(t')}$ (if no such index exists, we set $t' = -1$). For all indices smaller or equal to t', the statement follows from the monotonicity of the letter bounds u_i. We now show that also $\lambda_{i(t')} \leq \lambda_{i(t'+1)}$ holds. By definition of GreedyEqual it holds that

$$\begin{aligned}
\lambda_{(t'+1)} &= \frac{\ell - \Lambda(t'+\alpha)}{t - (t'+1)} \\
&= \frac{\ell - \Lambda(t'-1+\alpha) - \lambda_{i(t')}}{t - (t'+1)} \\
&\geq \frac{\ell - \Lambda(t'-1+\alpha) - \frac{\ell - \Lambda(t'-1+\alpha)}{t-t'}}{t - (t'+1)} \\
&= \frac{\ell - \Lambda(t'-1+\alpha)}{t-t'} \geq \lambda_{i(t')},
\end{aligned}$$

which holds with strict inequality if and only if $\lambda_{i(t')} = u_{i(t')} < \frac{\ell - \Lambda(t'-1+\alpha)}{t-t'}$. By analogous argumentation and a straightforward induction we get that

$$\lambda_{i(t'+1)} = \cdots = \lambda_{i(t-1)}.$$

If no city "wraps around", i.e., all $i(k)$ are distinct, then the statement follows directly by the order of the cities. In Theorem 5 we have argued that in order to wrap around, a city must be oversized, i.e., $\pi_i > \frac{1}{t}$. This concludes the proof. □

Lemma 7. *Independent of whether* GreedyEqual *succeeds, the following holds:*

(i) If GreedyEqual *selects the average at $x \in [0,t)$, then it selects the average at all $y \in [x, \lceil x \rceil)$ as well as all $y \in [x,t)$ with $y \equiv x \pmod 1$.*

(ii) The function $\Lambda(x)$ is non-decreasing on $[0,t)$.

Proof. (ii) We start by showing that the claim holds for each of the intervals $[t'-1,t')$ for $t' \in [t]$. We do so by induction over $t' \in [t]$. For the induction start, we consider the interval $[-1,0)$ where Λ is constant 0 and therefore the statement holds trivially.

Now, consider some $[t'-1,t')$ and assume that Λ is non-decreasing for the interval $[t'-2,t'-1)$. Let $\alpha, \beta \in [0,1)$ such that $\alpha < \beta$. Our goal is to show that $\Lambda(t'-1+\alpha) \leq \Lambda(t'-1+\beta)$. For every $k \in [t'-1]_0$ let $i(k)$ be the unique city with $\lambda_{i(k)}(k+\alpha) > 0$ and let $j(k)$ be the unique city with $\lambda_{j(k)}(k+\beta) > 0$. We write $\lambda_{i(k)}$ and $\lambda_{j(k)}$ instead of $\lambda_{i(k)}(k+\alpha)$ and $\lambda_{j(k)}(k+\beta) > 0$ from now on. We distinguish two cases:

Case 1: city $j(t'-1)$ receives their maximum amount of letter, i.e., $\lambda_{j(t'-1)} = u_{j(t'-1)}$. We claim that in this case $i(t'-1)$ also receives their maximum amount of letters. Formally,

$$u_{i(t'-1)} \leq u_{j(t'-1)} \leq \frac{\ell - \Lambda(t'-2+\beta)}{t-(t'-1)} \leq \frac{\ell - \Lambda(t'-2+\alpha)}{t-(t'-1)}$$

by induction hypothesis and therefore $\lambda_{i(t'-1)} = u_{i(t'-1)}$. Therefore, it follows directly from the induction hypothesis that

$$\begin{aligned}\Lambda_{i(t'-1)} &= \Lambda(t'-2+\alpha) + u_{i(t'-1)} \\ &\leq \Lambda(t'-2+\beta) + u_{j(t'-1)} = \Lambda(t'-1+\beta).\end{aligned}$$

Case 2: city $j(t'-1)$ does not receive their maximum amount of letter, i.e., $\lambda_{j(t'-1)} \neq u_{j(t'-1)}$. Then,

$$\begin{aligned}\Lambda(t'-1+\beta) &= \Lambda(t'-2+\beta) + \frac{\ell - \Lambda(t'-2+\beta)}{t-(t'-1)} \\ &\geq \Lambda(t'-2+\alpha) + \frac{\ell - \Lambda(t'-2+\alpha)}{t-(t'-1)} \\ &\geq \Lambda(t'-1+\alpha),\end{aligned}$$

where the first inequality follows from the fact that the stated expression is non-decreasing in Λ and Λ is monotone on $[t'-2,t'-1)$ by induction hypothesis. The second inequality follows from the upper bound on $\lambda_{i(t'-1)}$ in the definition of GreedyEqual.

We now move on to show that monotonicity also holds across intervals. Let $t' \in [t-1]_0$ and $\alpha, \beta \in [0,1)$ with $\alpha \leq \beta$. We aim to show that

$$\Lambda(t'-1+\beta) \leq \Lambda(t'+\alpha).$$

Given this statement, the more general statement for arbitrary $x, y \in [0,t)$ follows immediately by induction.

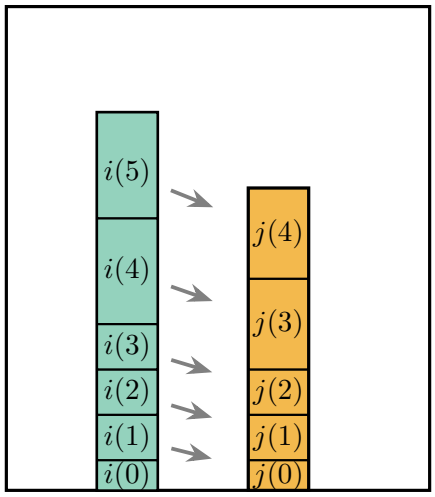


Figure 5: Situation in the second part of the proof of Lemma 7.

For every $k \in [t'-1]_0$ let $i(k)$ be the unique city with $\lambda_{i(k)}(k+\alpha) > 0$ and for every $k \in [t'-1]_0$ let $j(k)$ be the unique city with $\lambda_{j(k)}(k+\beta) > 0$. We refer to Figure 5 for an illustration of the situation and simply write $\lambda_{i(k)}$ and $\lambda_{j(k)}$ from now on. We prove the statement by showing that $\lambda_{i(k)} \geq \lambda_{j(k-1)}$ for all $k \in [t']$. Note that $i(0) \leq j(0) \leq \cdots \leq i(t'-1) \leq j(t'-1) \leq i(t')$. We distinguish two cases:

Case 1: city $i(k)$ receives their maximum amount of letter, i.e., $\lambda_{i(k)} = u_{j(k)}$. Then,

$$\lambda_{j(k-1)} \leq u_{j(k-1)} \leq u_{i(k)} = \lambda_{i(k)}.$$

Case 2: city $i(k)$ does not receive their maximum amount of letter, i.e., $\lambda_{i(k)} - \frac{\ell - \Lambda(k-1+\alpha)}{t-k}$. Then,

$$\begin{aligned}
\lambda_{i(k)} &= \frac{\ell - \Lambda(k-1+\alpha)}{t-k} \\
&\geq \frac{\ell - \Lambda(k-1+\beta)}{t-k} \\
&\geq \lambda_{j(k)} \\
&\geq \lambda_{j(k-1)},
\end{aligned}$$

where the first inequality follows from the monotonicity of Λ within the intervals, which we showed in the first part of the proof. The second inequality follows by the upper bound on $\lambda_{j(k)}$ enforced in the definition of GreedyEqual and the last inequality follows from the ex-post monotonicity which we showed in Theorem 6.

Therefore

$$\Lambda(t'+\alpha) > \sum_{k=1}^{t'} \lambda_{i(k)} \geq \sum_{k=0}^{t'-1} \lambda_{j(k)} = \Lambda(t'-1+\beta),$$

which concludes the proof.

(i) Let $i \in [n]$ and $x \in [0,t)$ be such that GreedyEqual sets $\lambda_i(x) = \frac{\ell - \Lambda(x-1)}{t - \lfloor x \rfloor}$. Let $y \in [x, \lceil x \rceil)$ and let j be the city that is active at point y (which might be x itself). Then, $u_i \leq u_j$ and

$\Lambda(x-1) \leq \Lambda(y-1)$ by statement (ii). Therefore

$$u_j \geq u_i \geq \frac{\ell - \Lambda(x-1)}{t - \lfloor x \rfloor} \geq \frac{\ell - \Lambda(y-1)}{t - \lfloor y \rfloor},$$

and therefore GreedyEqual selects the average at point y as well.

For the second part of the statement we refer to the proof of Theorem 6 which directly implies this statement.

□

Theorem 8. *Under Assumption 1, GreedyEqual is an additive 2-approximation for MinFeasibleCities.*

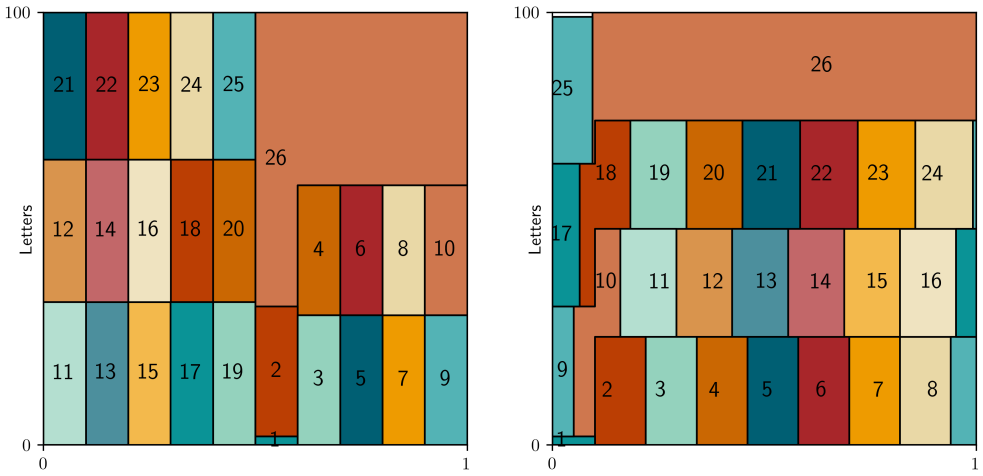
(second part). With the help of the above claim, we can now apply the scaling operation over all intervals. We overload notation when defining for each interval $[\alpha, \beta)$ and $k \in [t-1]_0$, we define the city $i(k)$ to be defined the unique city with the property that $\lambda_i(k)(k+\alpha) > 0$.

$$\begin{aligned}
\sum_{i \in [n]} w_i &= \sum_{i \in [n]} \frac{\pi_i \ell}{u_i} \\
&> \sum_{i \in [n]} \frac{\int_0^t \lambda_i(x) dx}{u_i} \\
&\geq \sum_{i \in [n]} \frac{\sum_{[\alpha,\beta) \in \mathcal{I}} \sum_{k=0}^{t-2} \lambda_i(k+\alpha)(\beta - \alpha)}{u_i} \\
&= \sum_{[\alpha,\beta) \in \mathcal{I}} (\beta - \alpha) \sum_{k=0}^{t-2} \frac{\lambda_{i(k)}(k+\alpha)}{u_{i(k)}} \\
&> \sum_{[\alpha,\beta) \in \mathcal{I}} (\beta - \alpha)(t-2) = t-2,
\end{aligned}$$

where the first inequality follows from the fact that GreedyEqual failed and therefore some cities received an area of less than their fair share. The second inequality follows by rewriting the integral and dropping the summand $k = t-1$. The equality follows from swapping the sums and the last inequality follows from the claim which we presented in the main part of the paper. As we have argued before, the above inequality implies that an optimal solution to MinFeasibleCities requires at least a budget of $t-1$ whenever GreedyEqual fails. Hence, GreedyEqual is an additive 2-approximation. □

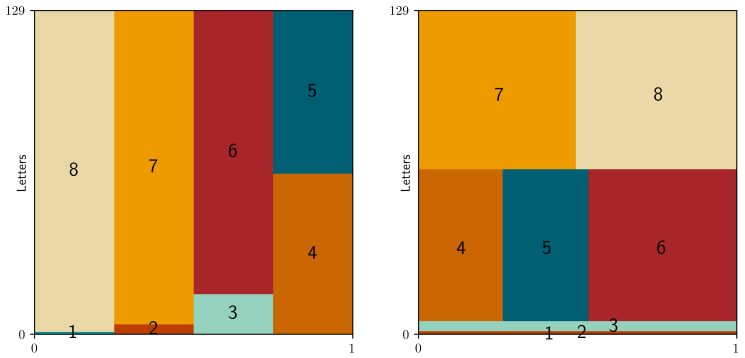
Theorem 9. *Even under Assumption 1, GreedyEqual is not an additive 1-approximation for MinFeasibleCities.*

Proof. We prove this statement by providing an example (Figure 6) for which a feasible solution with $t = 3$ exists while GreedyEqual indeed requires $t = 5$. Consider the instance with $\vec{\pi} = (\frac{2}{1000}, 9 \times \frac{30}{1000}, 10 \times \frac{33}{1000}, 5 \times \frac{34}{1000}, \frac{228}{1000})$ with $\vec{u} = 1000\pi$ and $\ell = 100$. There exists a solution with $t = 3$, visualized in Figure 6a. However, GreedyEqual fails for $t = 4$, as illustrated in Figure 6b. □



(a) Optimal solution with $t = 3$ (b) GreedyEqual failing for $t = 4$

Figure 6: Example instance showing that GreedyEqual is not a 1-approximation.



(a) Optimal solution with $t = 2$ (b) Buckets for $t = 5$

Figure 7: Example instance showing that Buckets is not a 2-approximation. $c = 1$, $z = 4$, $\pi = 516$, $\vec{\pi} = (\frac{1}{516}, \frac{4}{516}, \frac{16}{516}, \frac{64}{516}, \frac{65}{516}, \frac{113}{516}, \frac{125}{516}, \frac{128}{516})$, $\vec{u} = 516\vec{\pi} = (1, 4, 16, 64, 65, 113, 125, 128)$, $\ell = 129$

B Missing Proofs and Details of Section 5

Theorem 10. *For any targets and constant c, Buckets is not an additive c-approximation for MinFeasibleCities.*

Proof. Let $c \geq 1$ be a constant. We construct an instance which is feasible for $t = 2$, for which the bucket approach fails for any $t \leq 2 + c$. Define $z = c + 3$ and $\pi = 2z^z + z$. Consider an instance with $n = 2z$ cities and $\ell = \frac{\pi}{z}$ letters, where city i has size $\pi_i = \frac{z^{i-1}}{\pi}$ for $1 \leq i \leq \frac{n}{2}$ and $\pi_i = \frac{\frac{\pi}{z} - \pi_{n-i-1}}{\pi}$ for $z+1 \leq i \leq 2z$. Each cities maximum number of letters is given as $u_i = \pi_i \pi$. The optimal solution achieves $t = 2$ by choosing any pair of cities k and $n - k$ for $k \in [w]$ uniformly at random and assigning them their maximum number of letters. More formally, this solution is given as $\lambda_i(x) = u_i$ at positions $x \in [\frac{i-1}{z}, \frac{i}{z})$ for $1 \leq i \leq z$ and $\lambda_i(x) = u_i$ at positions $x \in [\frac{n+z-i}{z}, \frac{n+z-i+1}{z})$ for $z + 1 \leq i \leq 2z$. Figure 7a shows the optimal solution for $c = 2$.

However, BUCKETS fails for any $t \leq c+2$, independent of the choice of target letters. This is because any city $1 \leq i \leq z-1$ will end up in a single bucket, i.e. in the algorithm we have $i^\star = i = j$ in the first $\frac{n}{2}-1$ iterations, since $\sum_{k=i}^{i+1} \pi_k \ell = (z^{i-1}+z^i)\frac{\pi}{z} = (z^{i-2}+z^{i-1})\pi > z^{i-1}\pi = u_i$. Figure 7a shows the probablity distribution constructed by BUCKETS solution for $c=2$ and $t=5$. Thus, for any $t \leq z-1 = c+2$, BUCKETS will have $i < n$ after the loop and fails. □

Column Generation

We formulate finding the most proportional probability distribution as a linear program with a variable x_a for each t-bounded allocation $a \in A_t$, representing its probability.

$$\begin{aligned}
\text{minimize} \quad & \sum_{a \in A_t} x_a \varphi(a) \\
\text{subject to} \quad & \sum_{a \in A_t} x_a = 1, && (y) \\
& \sum_{a \in A_t} x_a a_i \geq \pi_i \ell \quad \text{for } i \in [n], && (y_i) \\
& x_a \in \mathbb{R} \quad \text{for } a \in A_t, \\
& x_a \geq 0, \quad \text{for } a \in A_t.
\end{aligned}$$

To approximate or find an optimal solution for the LP, we first formulate the dual LP with variables y and y_i for each city $i \in [n]$.

$$\begin{aligned}
\text{maximize} \quad & y + \sum_{i \in [n]} \pi_i \ell y_i \\
\text{subject to} \quad & y + \sum_{i \in [n]} a_i y_i \leq \varphi(a) \quad \text{for } a \in A_t, && (x_a) \\
& y \in \mathbb{R}, \\
& y_i \geq 0, \quad \text{for } i \in [n].
\end{aligned}$$

We start with any ex-ante fair probability distribution over A_t and solve the above LP with only the constraints corresponding to allocations a with positive probability. We then iteratively compute an allocation that we can add to decrease the disproportionality measure.

Given a solution y, y_i for a (partial version of) the dual above, we define the separation oracle as a mixed integer program with variables a_i and auxiliary binary variables z_i for each city $i \in [n]$:

$$\begin{aligned}
\text{maximize} \quad & y + \sum_{i \in [n]} a_i y_i - \sum_{i \in [n]} \left| \frac{a_i}{\tau_i} - 1 \right| \\
\text{subject to} \quad & \sum_{i \in [n]} a_i = \ell, \\
& \sum_{i \in [n]} z_i \leq t, \\
& a_i \leq u_i, && \text{for } i \in [n], \\
& a_i \leq z_i \ell && \text{for } i \in [n],
\end{aligned}$$

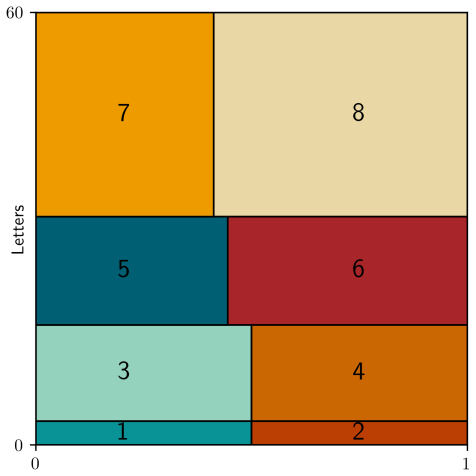


Figure 8: Probability distribution constructed by BUCKETS for Example 1, with $t = 4$ and square root target letters.

$$z_i \in \{0, 1\}, \qquad \text{for } i \in [n],$$
$$a_i \geq 0, \qquad \text{for } i \in [n].$$

Note that the objective contains an absolute value, which can be linearized using standard ILP techniques. We implemented this separation oracle using Gurobi 12 (under academic license).

Bucket Approach

Below we provide a formal description of BUCKETS which creates in each while loop a bucket containing cities i to i^*. Note that the second part of the pseudocode (starting from the for loop) is only there to draw a picture similar to those of GREEDYEQUAL. Each bucket is filled, until its height would be higher than the letter upper bound of its smallest city or the bucket target width would be larger than 1. Note, that the stop condition in the algorithm implicitly rescales the remaining municipalities target width to a total of $t - j + 1$

---

procedure BUCKETS($\vec{\pi}, \vec{u}, t, \vec{\tau}$)
  $i \leftarrow 1$, $j \leftarrow 1$
  while $j \leq t$ and $i \leq n$ **do**
    $i^\star \leftarrow \max\left\{ i' \in [n] \;\middle|\; \sum_{k=i}^{i'} \pi_k \ell \leq u_i \text{ and } (t - j + 1) \sum_{k=i}^{i'} \tau_k \leq \sum_{k=i}^{n} \tau_k \right\}$
    $x \leftarrow j - 1$
    for $k = i, \ldots, i^\star$ **do**
      $h \leftarrow \sum_{k=i}^{i^\star} \pi_k \ell$
      $\lambda_k(z) \leftarrow h$ for $z \in [x, x + \frac{\pi_k \ell}{h})$
      $x \leftarrow x + \frac{\pi_k \ell}{h}$
    $j \leftarrow j + 1$, $i \leftarrow i^\star + 1$
  if $i < n$ **then return** "fail"

C Missing Details of Section 6

In this section we explain in more detail how we distribute the total number of letters ℓ and the number of cities to be selected $t = 80$ over the 42 groups of cities. For each group, we want a number of letters ℓ_G and a number of cities to contact t_G to apply one of the methods presented in Sections 4 and 5. Given that the total number of letters is large ($\ell = 20\,000$), we can apply randomized rounding to these fair shares without introducing significant deviations.

The distribution of t over the groups on the other hand is not determined by ex-ante fairness. However, there are upper and lower bounds on the number of cities we select from each group. For each group, t_G must be at least the minimum number $t_G^{\min}$ that the corresponding algorithm requires for the group (note, that this will always be at least 1). On the other hand, t_G can not be larger than the number of cities n_g in a group, which is mostly relevant for the groups with only one or two cities (compare Table 2). Under these constraints, we then try to apportion t in a way, such that similarly sized cities receive similar number of letters if selected, across groups. First, we compute the *global* target letters τ_i^{glob} for each city with respect to the total number of letters $\ell = 20\,000$ and $t = 80$. Since we will have to recompute these targets within each group G, depending on the value we assign to t_G and we want the recomputed targets to be close to the global ones, we try to keep t_G close to the sum the sum of (global) target widths within a group, limiting the amount of rescaling required. To achieve this, define the parametrized target width t_G^γ of a group as the rounded sum of (global) target widths of its cities, bounded by the aforementioned bounds.

$$t_G^\gamma = \max\left(\min\left(\left\lceil \gamma \sum_{i \in N_G} \frac{\pi_i \ell}{t_G^{\text{glob}}} \right\rceil, n_g\right), t_G^{\min}\right). \tag{9}$$

Then, find a value γ, such that $\sum_{G \in \mathcal{G}} t_G^\gamma = 80$ and assign $t_G = t_G^\gamma$ to each group G. This process can be described as running Adam's apportionment method [Balinski and Young, 2001] on the global target widths of the groups, while enforcing their upper and lower bounds. In principle, we could use any rounding function in Equation (9), however rounding up seems like a natural choice, since it ensures that each group receives at least one, slightly reducing the bias that is introduced by the values of $t_G^{\min}$.

For the experimental results and an evaluation of this method, see Appendix D.

D Experimental Results

In this section we present the results of our experiments in more detail. All experiments were run on a machine with specifications as indicated in Table 1. The total running time of GreedyEqual and Buckets was below one minute, while ColumnGeneration took several hours to compute.

Results Within Groups

We compute the results of GreedyEqual, ColumnGeneration and Buckets for each of the 42 groups of cities and present the results in Figures 9 to 50a.

Note, that since the choice of t_G depends on the method and target letter function, t_G can differ between methods. In addition to the output probability distributions, we also visualize how well each method aligns with the target letters. We refer to the explanations in Section 5 on how to read these figures. We use the square-root target function $f(x) = \sqrt{x}$ for ColumnGeneration and Buckets, and also evaluate GreedyEqual with respect to the target function $f(x) = \ell$, even though it does not take any targets as input.

Hardware Model	HP ZBook Power 15.6 inch G9 Mobile Workstation PC
Memory	16.0 GiB
Processor	12th Gen Intel® Core™ i7-12700H × 20
Graphics	Mesa Intel® Graphics (ADL GT2)
Disk Capacity	2.5 TB
OS Name	Ubuntu 22.04.4 LTS

Table 1: Computer specifications

For the groups of medium and large cities GreedyEqual mostly achieves its goal of assigning all cities the same number of letters. For the groups of small cities this is not possible, thus some of the larger cities can receive variable numbers of letters. Even though the assumption from Theorem 6 on city sizes is not met in all groups, there are no ex-post monotonicity violations in any of the probability distributions constructed by GreedyEqual.

ColumnGeneration iteratively computes the probability distribution, solving a mixed integer program in each iteration. In practice, the number of iterations needed until an optimal solution is found is in the hundreds for (non-trivial) groups of medium and large cities and in the thousands for the small ones. While perfectly matching the targets is not always possible (see for example Figures 30 and 44), ColumnGeneration comes very close for most of the groups (e.g., Figures 14 and 31). On the downside, ColumnGeneration often violates ex-post monotonicity and the number of letters sent to similarly sized cities can fluctuate, resulting in the spikes e.g. in Figure 44. The former could potentially be fixed, by restricting the separation oracle to monotone allocations, while the latter might be reduced by penalizing larger deviations more in the objective function.

Buckets groups cities of equal size and assigns them the same number of letters upon selection, trivially satisfying the binary outcome property. This often works well to approximate the target letters, as can be seen for example in Figures 14 and 23. The lower the value of t_G the coarser the approximation will get. Buckets works less well for groups which have municipalities with a very low letter upper bound u_i, as this forces all municipalities in the same bucket to receive the same number of letters upon selection (see for example Figures 11 and 50). Again, this issue worsens for low values of t_G. Similarly to GreedyEqual, even though ex-post monotonicity is not theoretically guaranteed, there are no monotonicity violations on our data. An additional advantage of the probability distributions returned by Buckets is that each bucket can be sampled independently. This reduces correlations and reshuffles the combinations of municipalities selected across multiple samplings.

Results Across Groups

Apart from the sampling methods for each group, we also proposed a way to initially assign the number of cities to be selected from each group, with the aim of keeping the target letters for equally sized cities close across groups. Figures 51 to 54 compare the global and local target functions and visualize how well ColumnGeneration and Buckets meet these targets. Figures 51 and 52 restrict their view to the targets for small cities. The global targets are drawn as a thick black line and the local targets of the groups are shown as thinner, colored lines. Additionally, each scatter point represents one city with their size on the x-axis and the expected number of letters they receive if selected on the y-axis. We can see that most local targets are fairly close to each other, with some notable exceptions. ColumnGeneration is generally quite close to the targets (as we

have seen in the previous section), while BUCKETS approximates the targets function through a series of horizontal lines.

Figures 53 and 54 compare the local target functions of all groups (note the log scales). Here, we observe a fairly large number of groups with constant local targets much lower than the global ones. These are exactly the groups that were assigned $t_G = 1$ in the apportionment step. When only allowed to choose one city, there is only one way to sample: select a city with probability proportional to size and assign it the full number of letters ℓ_G. The local target function reflects this and since these local targets are essentially independent of the target function f, the global and local targets can be arbitrarily far apart. This affects mostly the groups of medium and large, which is due to the target function and the resulting apportionment, which implicitly favors groups with small cities. An extreme example of this are the large cities of Saarland (compare Table 2). The cities have a joint global target width of 0.180371, but they must receive at least $t_G = 1$, resulting in local targets much lower than the global targets.

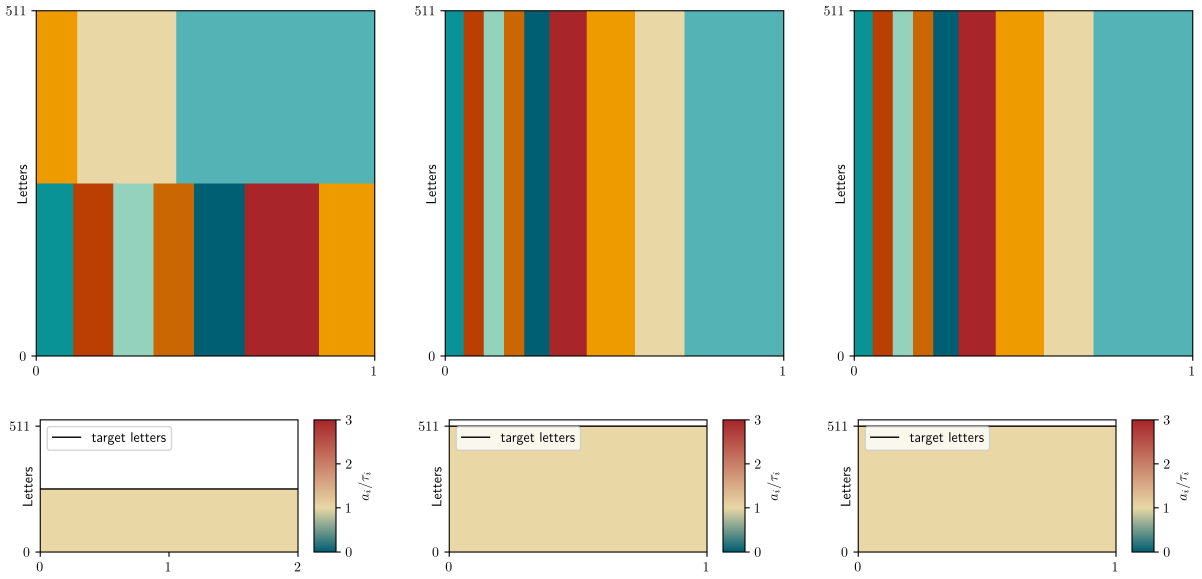
Note, that this is less an effect of the apportionment method and more of the choice of target function. Choosing a target function with slower growth or lowering the thresholds that define medium and large cities, could potentially increase the target width of these groups and consequently bring the local targets closer to the global ones.

Other Data

While we evaluated our methods for the large nation-wide citizen assemblies in Germany, there are other similar projects to which they could be applied. As an example, we applied our methods to the whole state of Baden-Württemberg with $t = 11$ and $\ell = 2674$. The results can be seen in Figure 55.

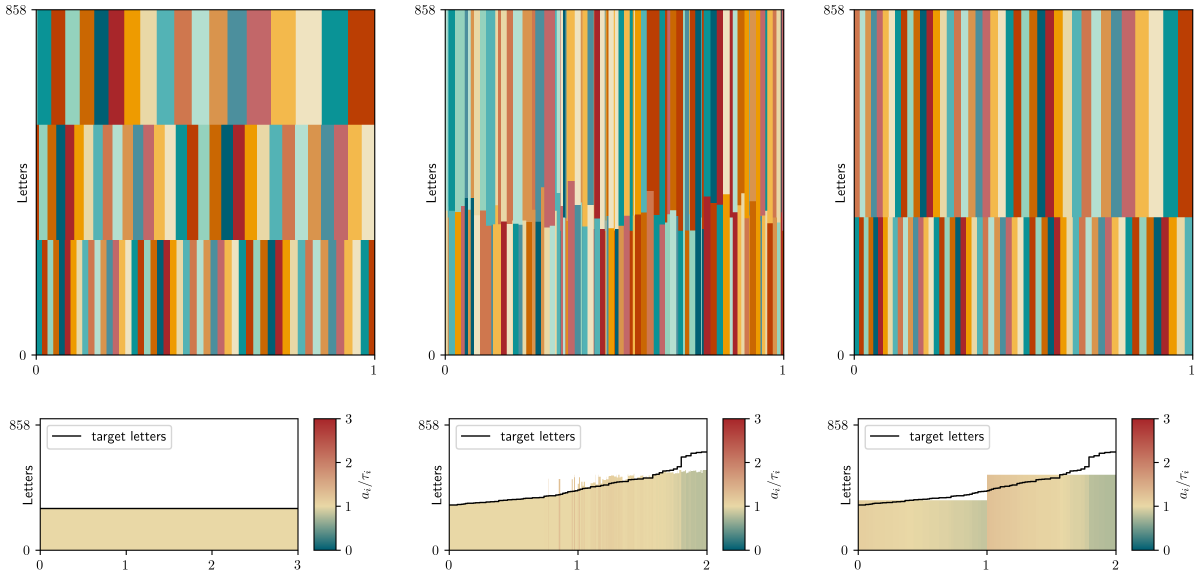
Table 2: Results of the apportionment. $|G|$ is the number of cities in the group, t_G^{GE}, t_G^{CG} and t_G^{B} are the values of t_G for GreedyEqual, ColumnGeneration and Buckets, respectively.

Group	Population	π_i	$\lvert G\rvert$	ℓ_G	t_G^{GE}	t_G^{CG}	t_G^{B}
Baden-Württemberg (Large)	2158197	0.0256	9	511	2	1	1
Baden-Württemberg (Medium)	3619302	0.0429	98	858	3	2	2
Baden-Württemberg (Small)	5502758	0.0652	994	1305	4	6	6
Bayern (Large)	3010827	0.0357	8	714	2	1	1
Bayern (Medium)	2326541	0.0276	67	551	2	1	1
Bayern (Small)	8032025	0.0952	1981	1905	6	10	10
Berlin (Large)	3755251	0.0445	1	891	1	1	1
Brandenburg (Large)	185750	0.0022	1	44	1	1	1
Brandenburg (Medium)	940363	0.0111	27	223	1	1	1
Brandenburg (Small)	1447022	0.0172	385	343	2	2	2
Bremen (Large)	684864	0.0081	2	162	1	1	1
Hamburg (Large)	1892122	0.0224	1	449	1	1	1
Hessen (Large)	1658130	0.0197	6	393	2	1	1
Hessen (Medium)	1805347	0.0214	53	428	2	1	1
Hessen (Small)	2927883	0.0347	362	694	2	3	3
Mecklenburg-Vorpommern (Large)	209920	0.0025	1	50	1	1	1
Mecklenburg-Vorpommern (Medium)	396680	0.0047	8	94	1	1	1
Mecklenburg-Vorpommern (Small)	1021778	0.0121	716	242	2	2	2
Niedersachsen (Large)	1588358	0.0188	8	377	2	1	1
Niedersachsen (Medium)	2974786	0.0353	86	705	2	2	2
Niedersachsen (Small)	3577098	0.0424	847	848	3	4	4
Nordrhein-Westfalen (Large)	8438299	0.1000	30	2001	6	2	2
Nordrhein-Westfalen (Medium)	7369437	0.0874	182	1747	5	3	3
Nordrhein-Westfalen (Small)	2331380	0.0276	184	553	2	2	2
Rheinland-Pfalz (Large)	723508	0.0086	5	172	1	1	1
Rheinland-Pfalz (Medium)	690561	0.0082	17	163	1	1	1
Rheinland-Pfalz (Small)	2745081	0.0325	2279	651	3	6	6
Saarland (Large)	181959	0.0022	1	43	1	1	1
Saarland (Medium)	275178	0.0033	8	65	1	1	1
Saarland (Small)	535529	0.0063	43	127	1	1	1
Sachsen (Large)	1427967	0.0169	3	338	1	1	1
Sachsen (Medium)	723183	0.0086	21	172	1	1	1
Sachsen (Small)	1935002	0.0229	394	458	2	3	3
Sachsen-Anhalt (Large)	481447	0.0057	2	114	1	1	1
Sachsen-Anhalt (Medium)	708172	0.0084	22	168	1	1	1
Sachsen-Anhalt (Small)	997024	0.0118	194	237	1	2	1
Schleswig-Holstein (Large)	465812	0.0055	2	110	1	1	1
Schleswig-Holstein (Medium)	753307	0.0089	20	179	1	1	1
Schleswig-Holstein (Small)	1734151	0.0206	1082	411	3	3	4
Thüringen (Large)	326160	0.0039	2	77	1	1	1
Thüringen (Medium)	692661	0.0082	20	164	1	1	1
Thüringen (Small)	1108025	0.0131	583	263	2	2	2



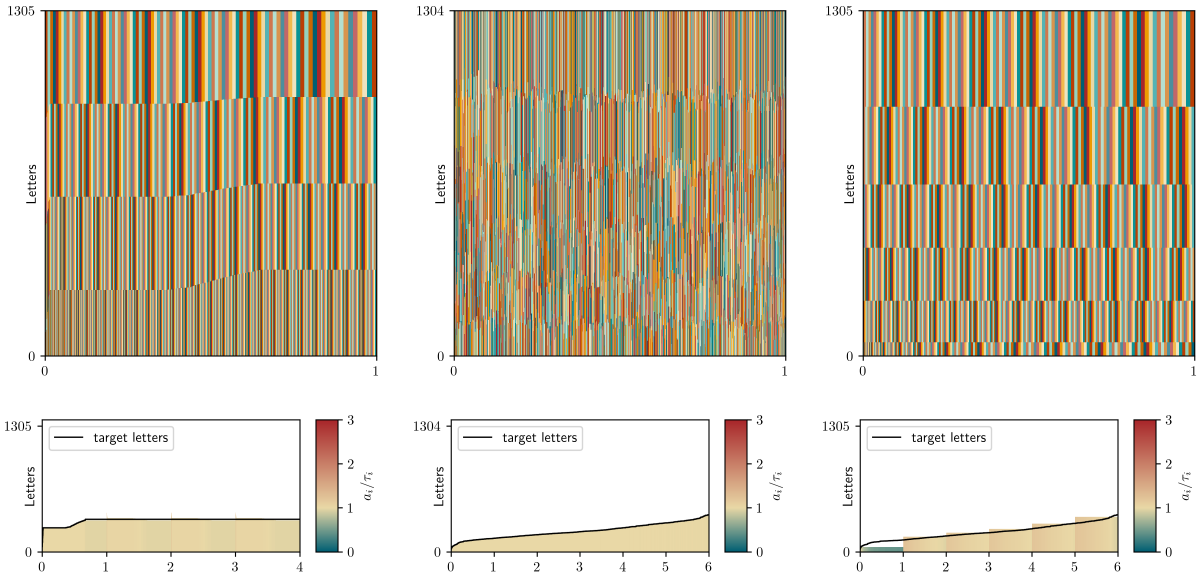
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 9: Large municipalities of Baden-Württemberg ($\ell_G = 511$)



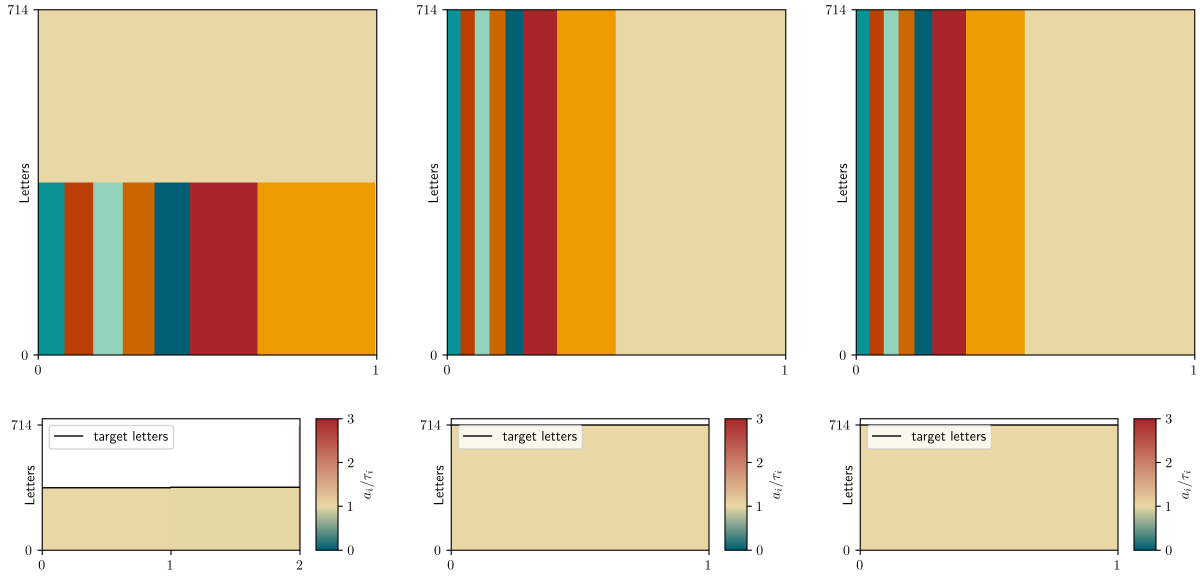
(a) GreedyEqual ($t_G = 3$) (b) ColumnGeneration ($t_G = 2$) (c) Buckets ($t_G = 2$)

Figure 10: Medium municipalities of Baden-Württemberg ($\ell_G = 858$)



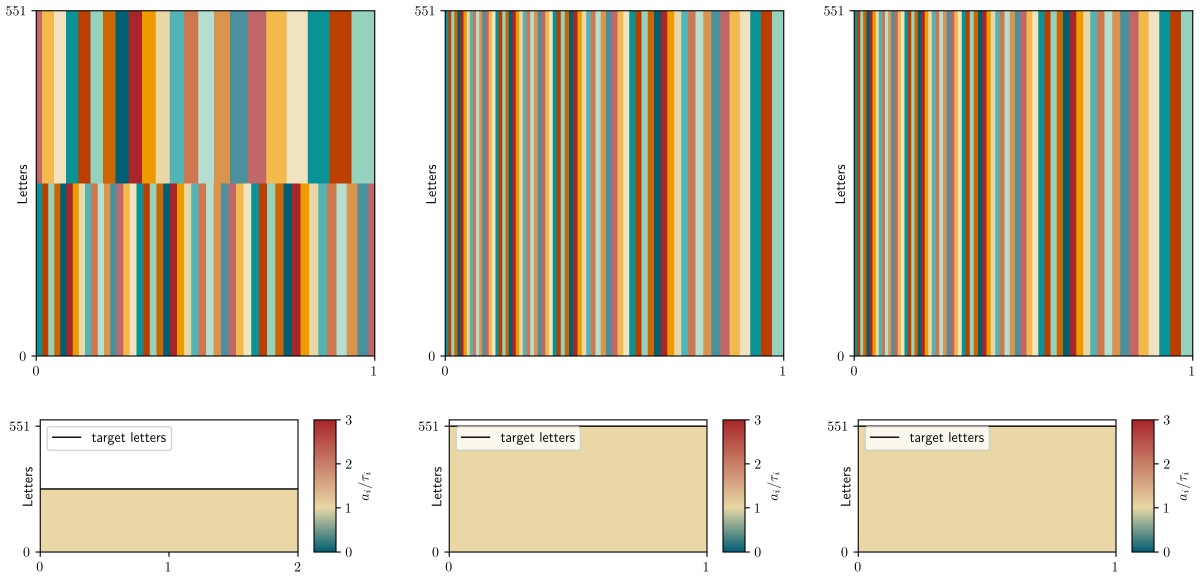
(a) GreedyEqual ($t_G = 4$) (b) ColumnGeneration ($t_G = 6$) (c) Buckets ($t_G = 6$)

Figure 11: Small municipalities of Baden-Württemberg ($\ell_G = 1305$)



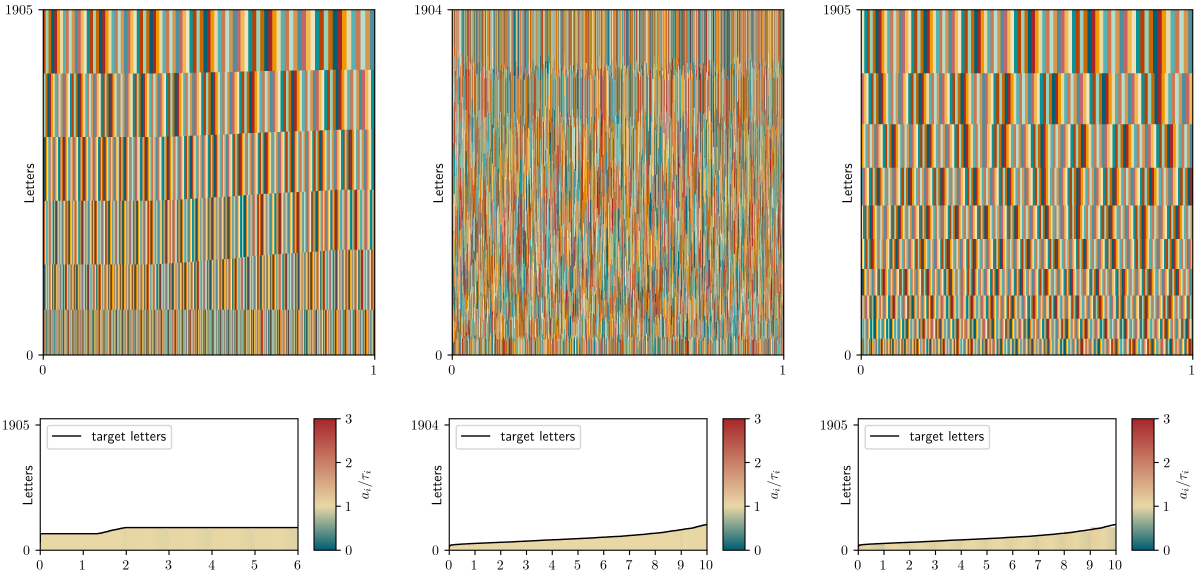
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 12: Large municipalities of Bayern ($\ell_G = 714$)



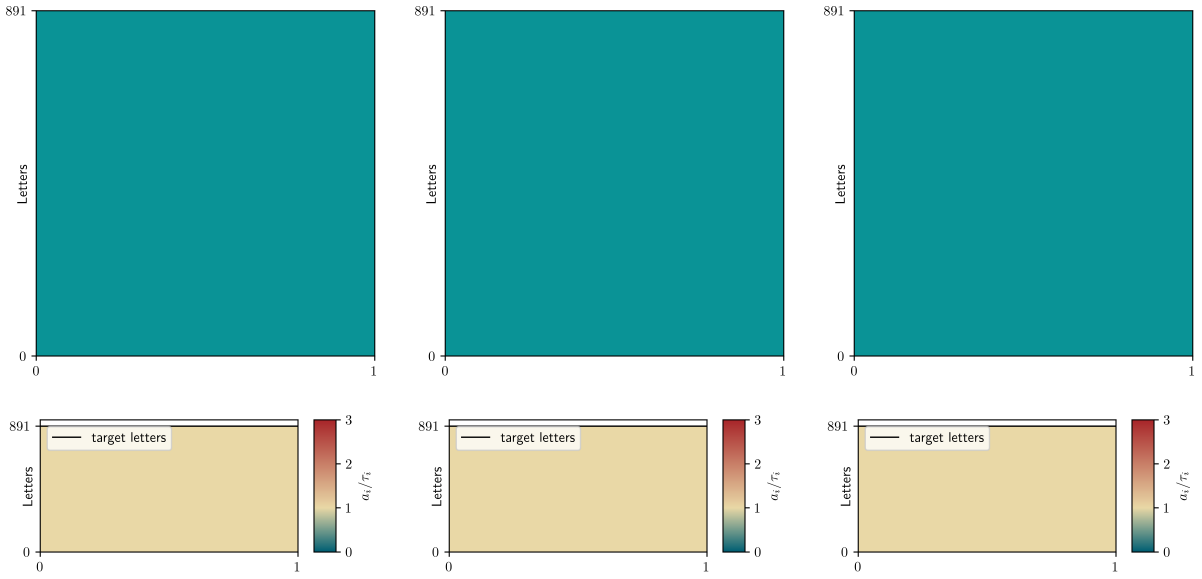
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 13: Medium municipalities of Bayern ($\ell_G = 551$)



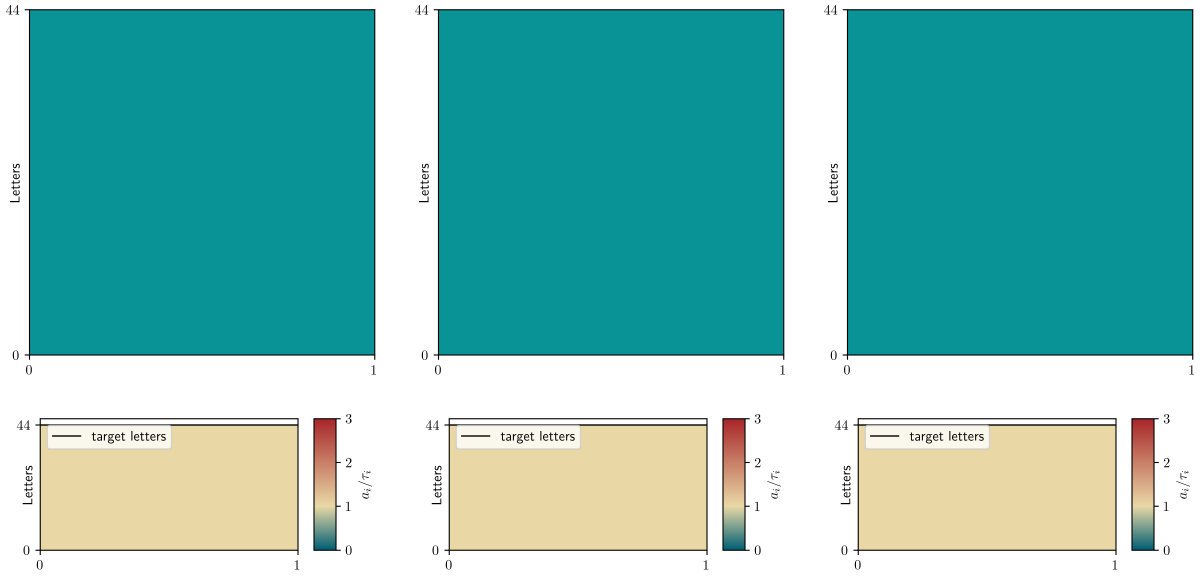
(a) GreedyEqual ($t_G = 6$) (b) ColumnGeneration(t_G=10) (c) Buckets ($t_G = 10$)

Figure 14: Small municipalities of Bayern ($\ell_G = 1905$)



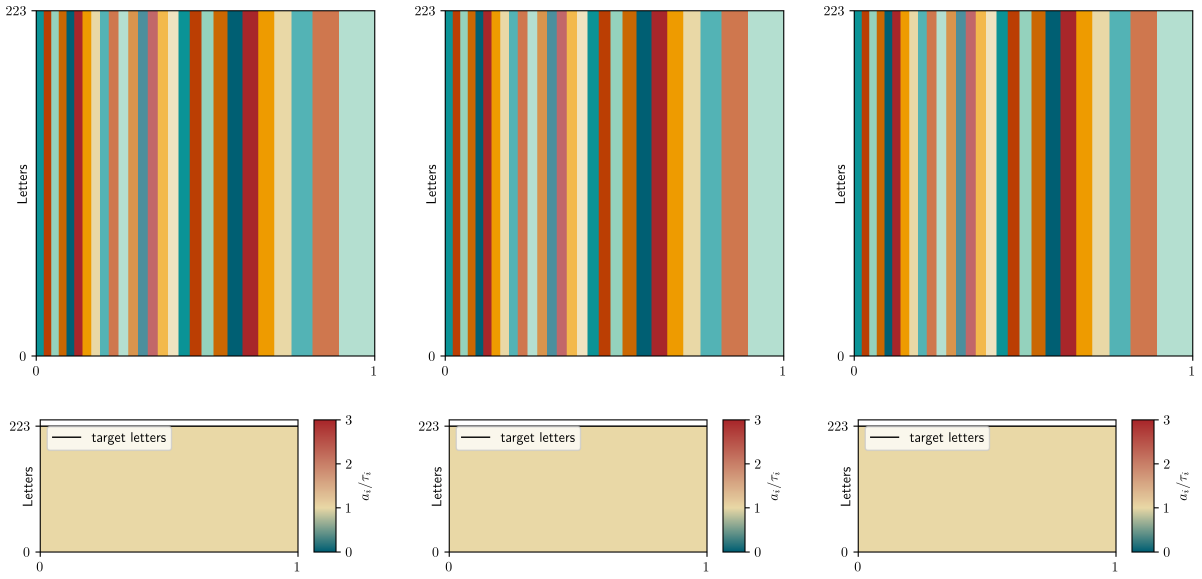
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 15: Large municipalities of Berlin ($\ell_G = 891$)



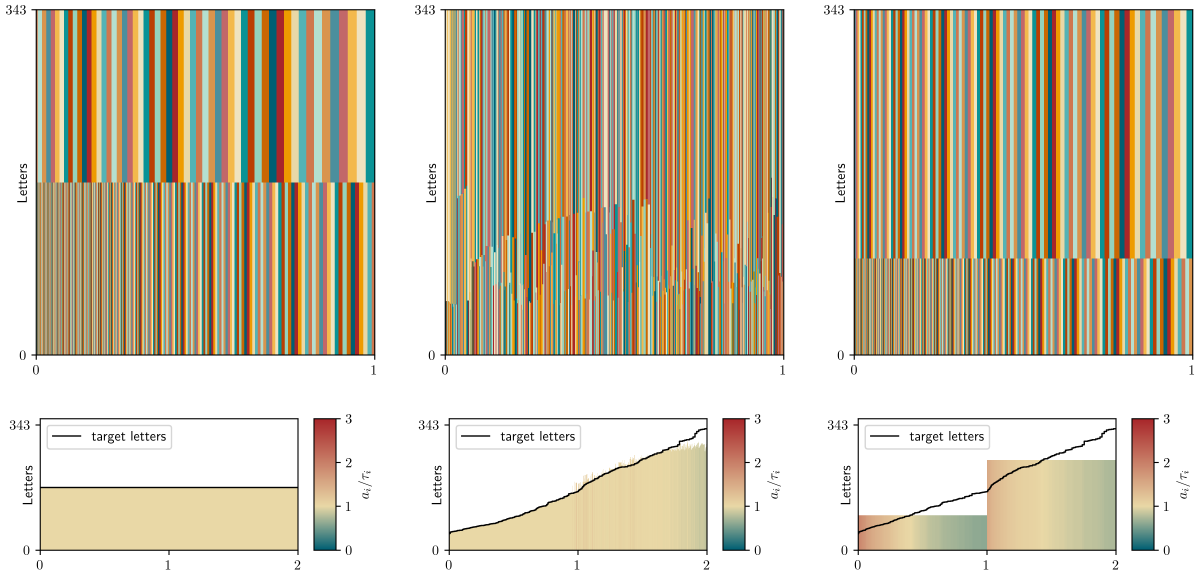
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 16: Large municipalities of Brandenburg ($\ell_G = 44$)



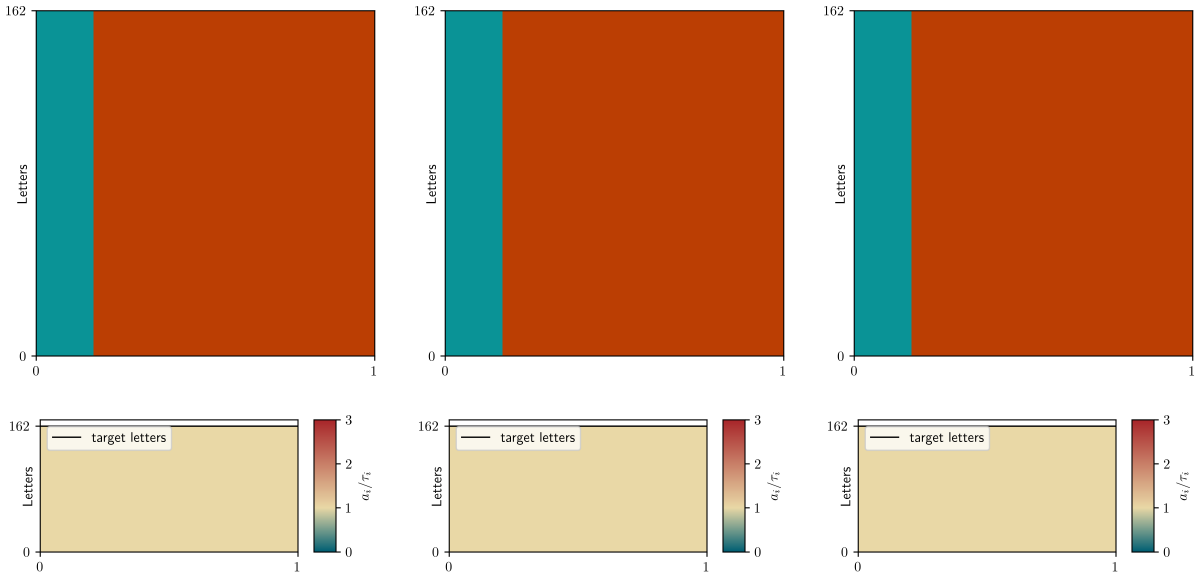
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 17: Medium municipalities of Brandenburg ($\ell_G = 223$)



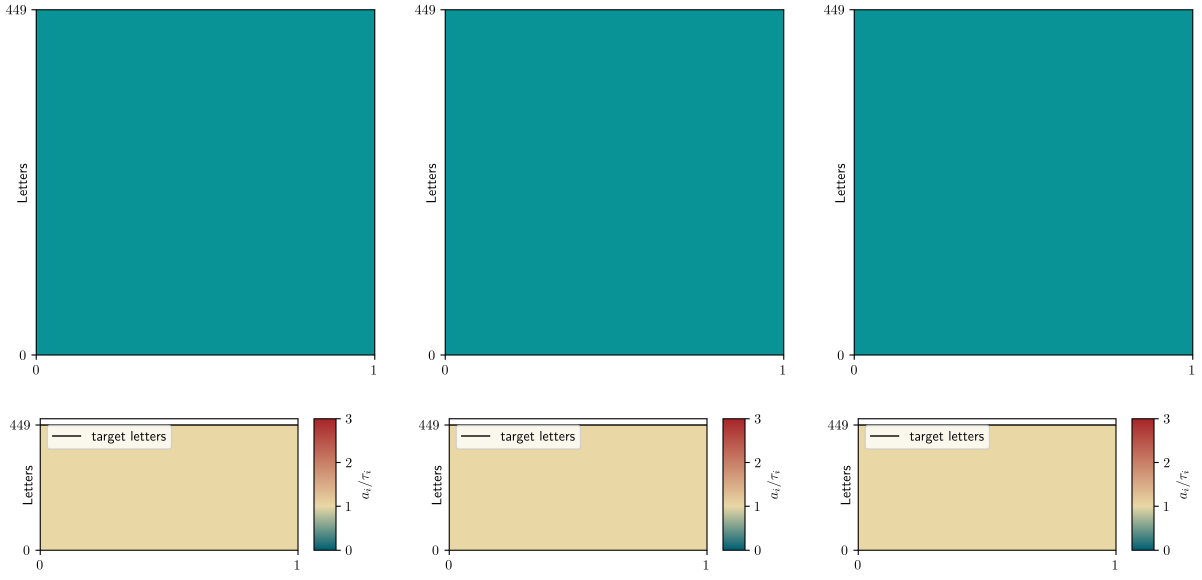
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 2$) (c) Buckets ($t_G = 2$)

Figure 18: Small municipalities of Brandenburg ($\ell_G = 343$)



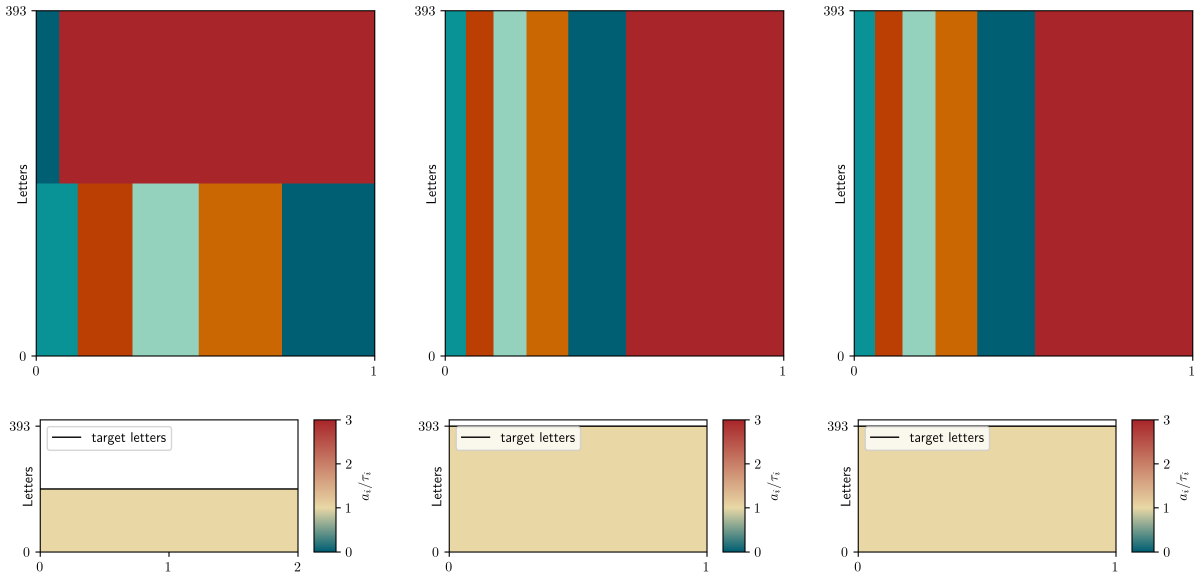
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 19: Large municipalities of Bremen ($\ell_G = 162$)



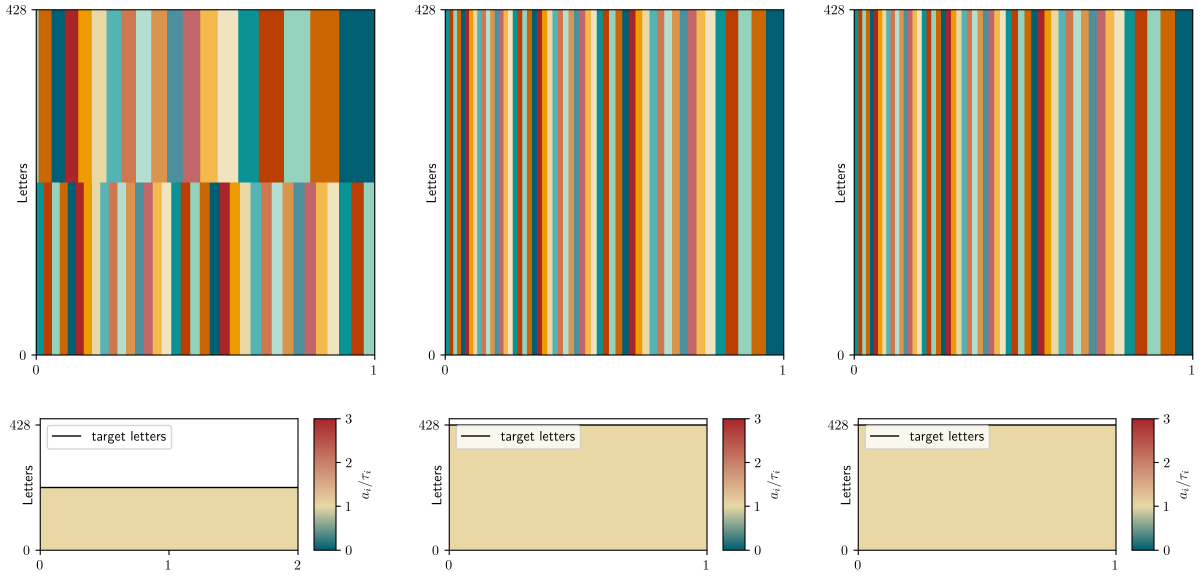
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 20: Large municipalities of Hamburg ($\ell_G = 449$)



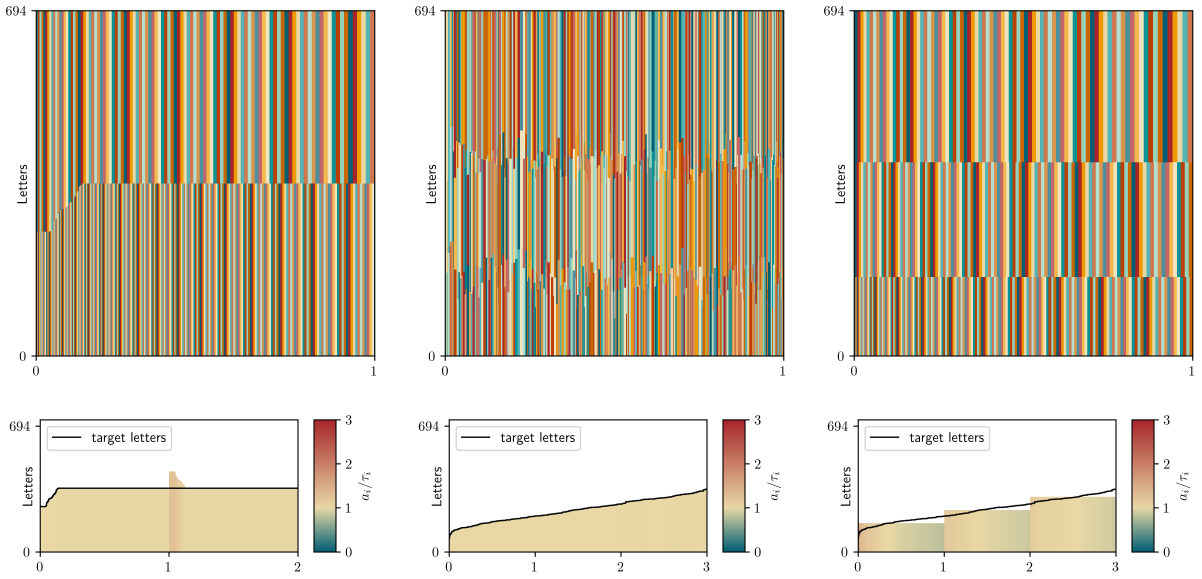
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 21: Large municipalities of Hessen ($\ell_G = 393$)



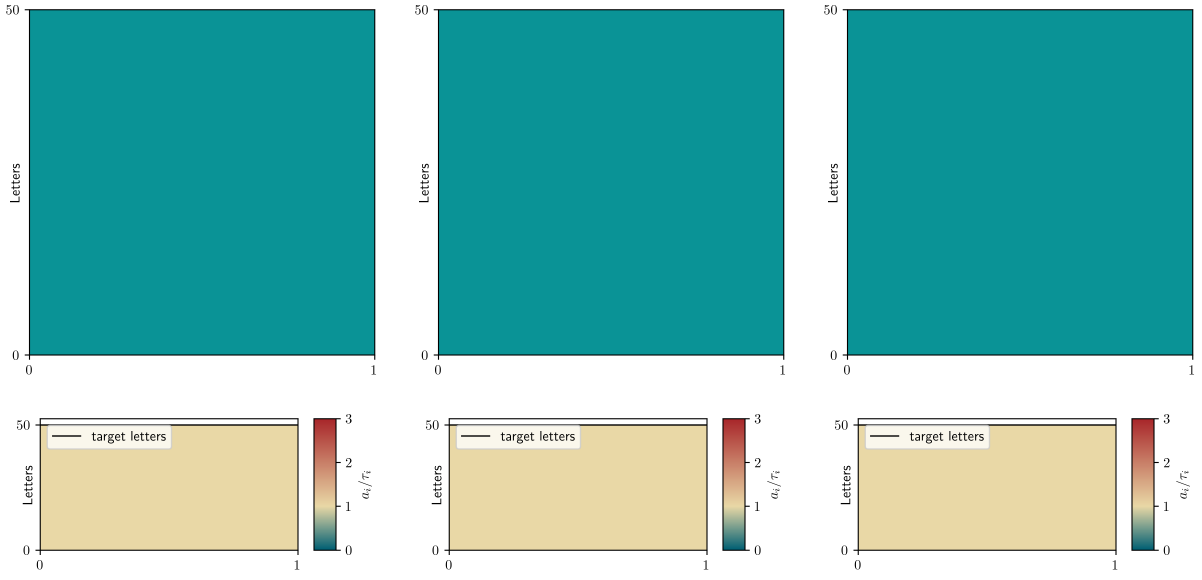
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 22: Medium municipalities of Hessen ($\ell_G = 428$)



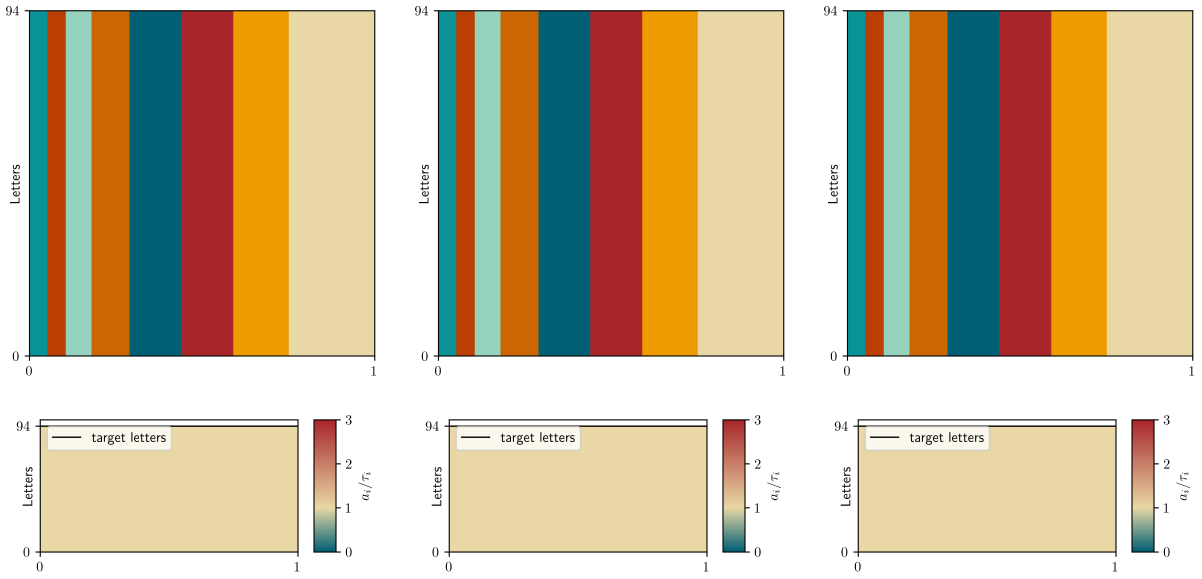
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 3$) (c) Buckets ($t_G = 3$)

Figure 23: Small municipalities of Hessen ($\ell_G = 694$)



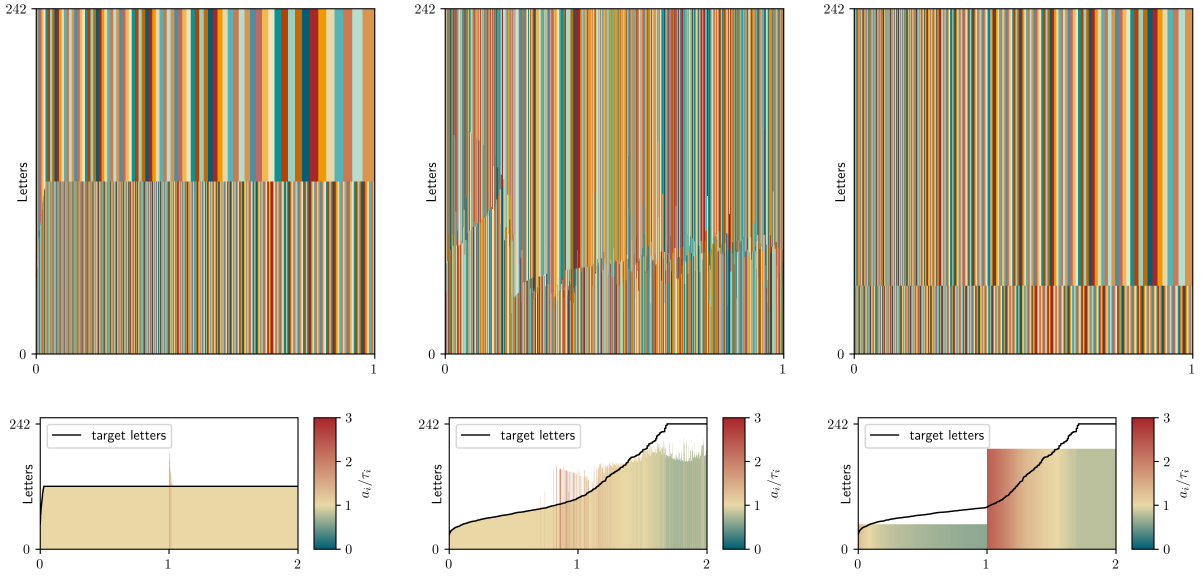
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 24: Large municipalities of Mecklenburg-Vorpommern ($\ell_G = 50$)



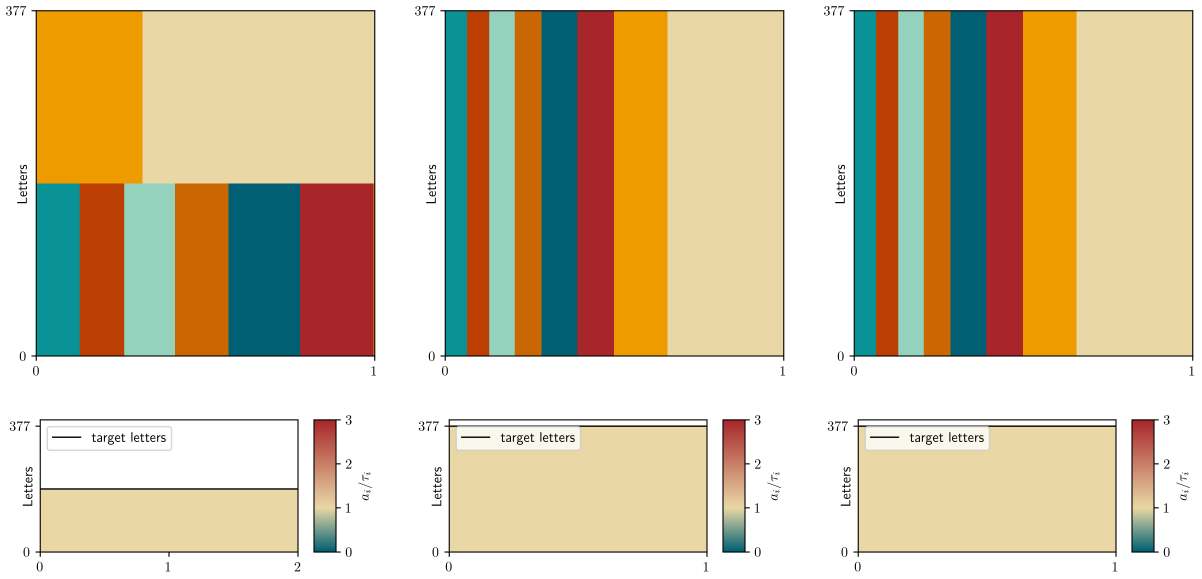
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 25: Medium municipalities of Mecklenburg-Vorpommern ($\ell_G = 94$)



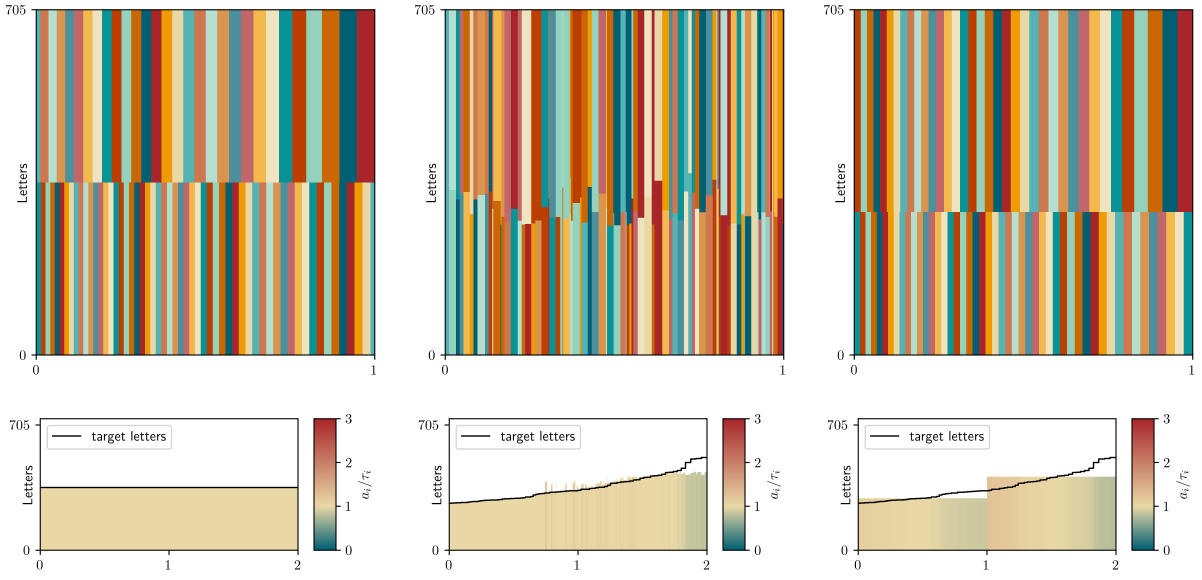
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 2$) (c) Buckets ($t_G = 2$)

Figure 26: Small municipalities of Mecklenburg-Vorpommern ($\ell_G = 242$)



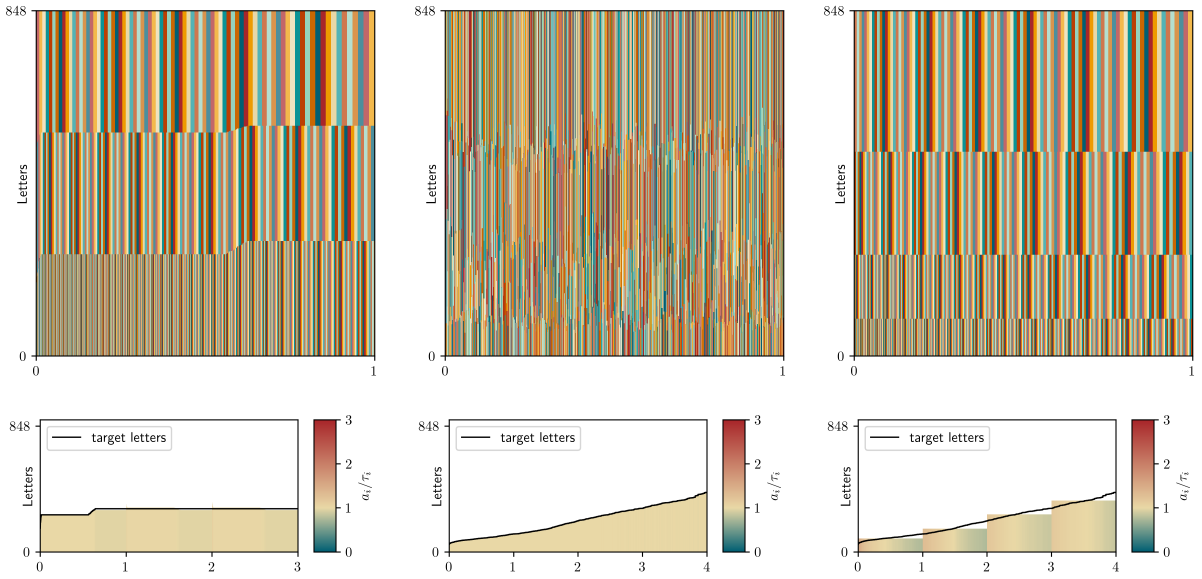
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 27: Large municipalities of Niedersachsen ($\ell_G = 377$)



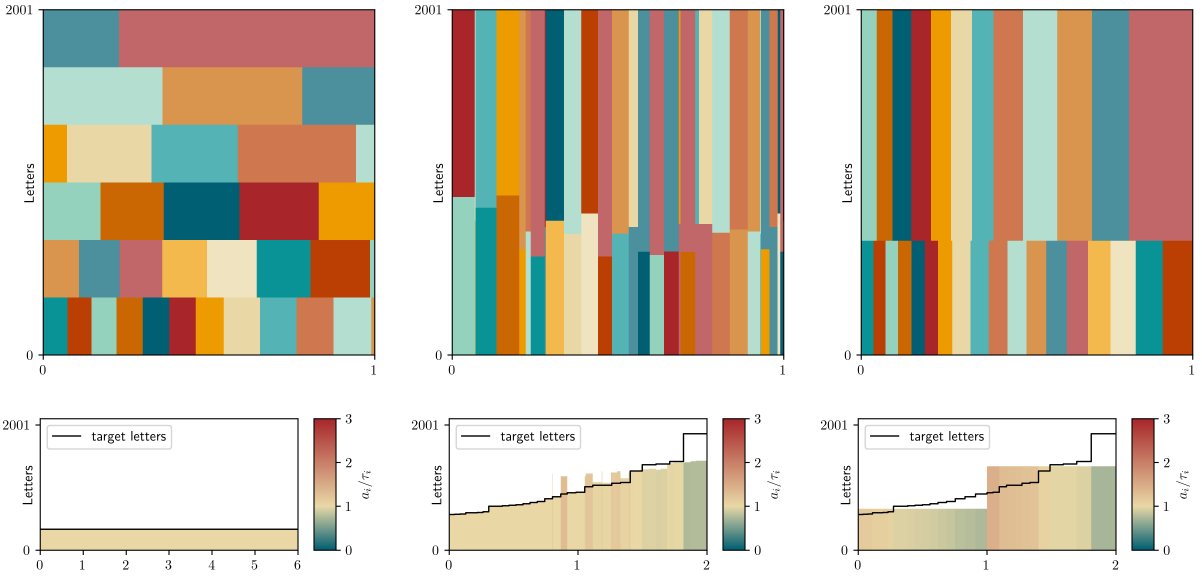
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 2$) (c) Buckets ($t_G = 2$)

Figure 28: Medium municipalities of Niedersachsen ($\ell_G = 705$)



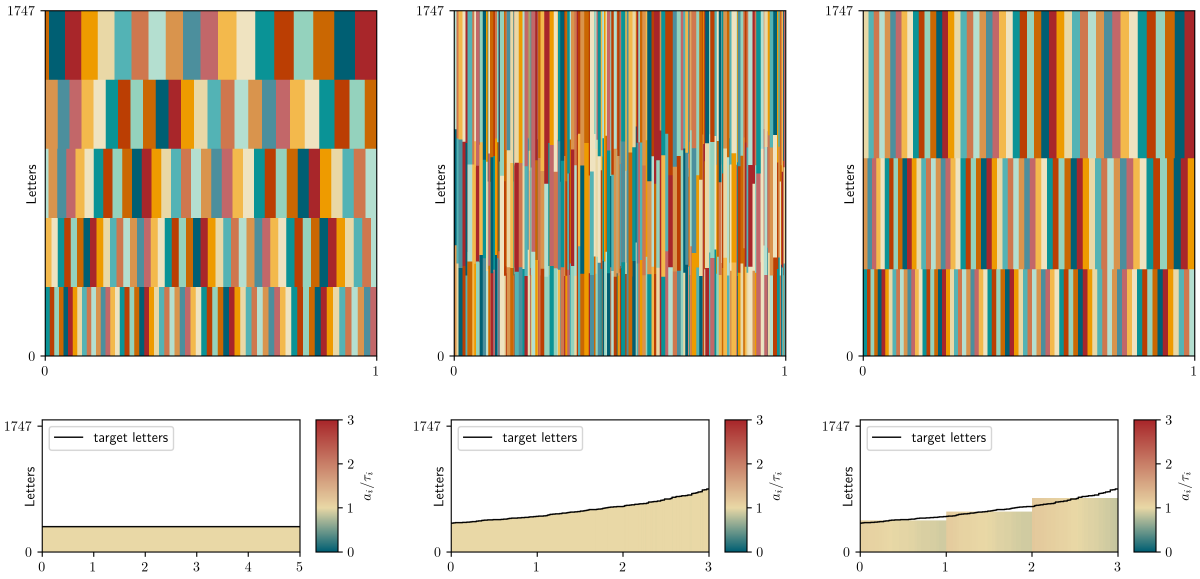
(a) GreedyEqual ($t_G = 3$) (b) ColumnGeneration ($t_G = 4$) (c) Buckets ($t_G = 4$)

Figure 29: Small municipalities of Niedersachsen ($\ell_G = 848$)



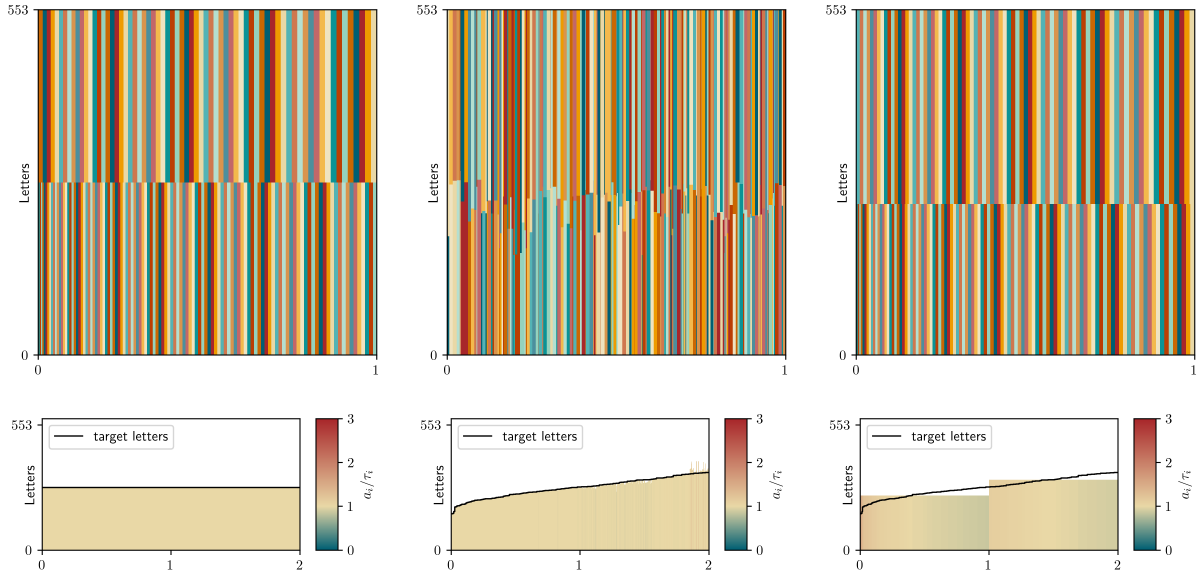
(a) GreedyEqual ($t_G = 6$) (b) ColumnGeneration ($t_G = 2$) (c) Buckets ($t_G = 2$)

Figure 30: Large municipalities of Nordrhein-Westfalen ($\ell_G = 2001$)



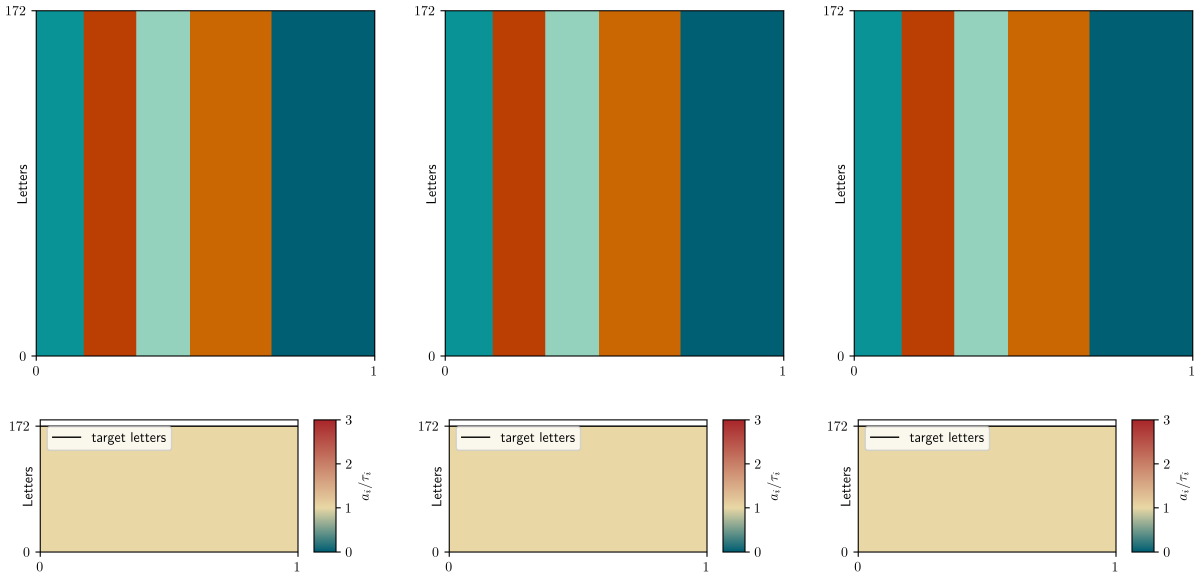
(a) GreedyEqual ($t_G = 5$) (b) ColumnGeneration ($t_G = 3$) (c) Buckets ($t_G = 3$)

Figure 31: Medium municipalities of Nordrhein-Westfalen ($\ell_G = 1747$)



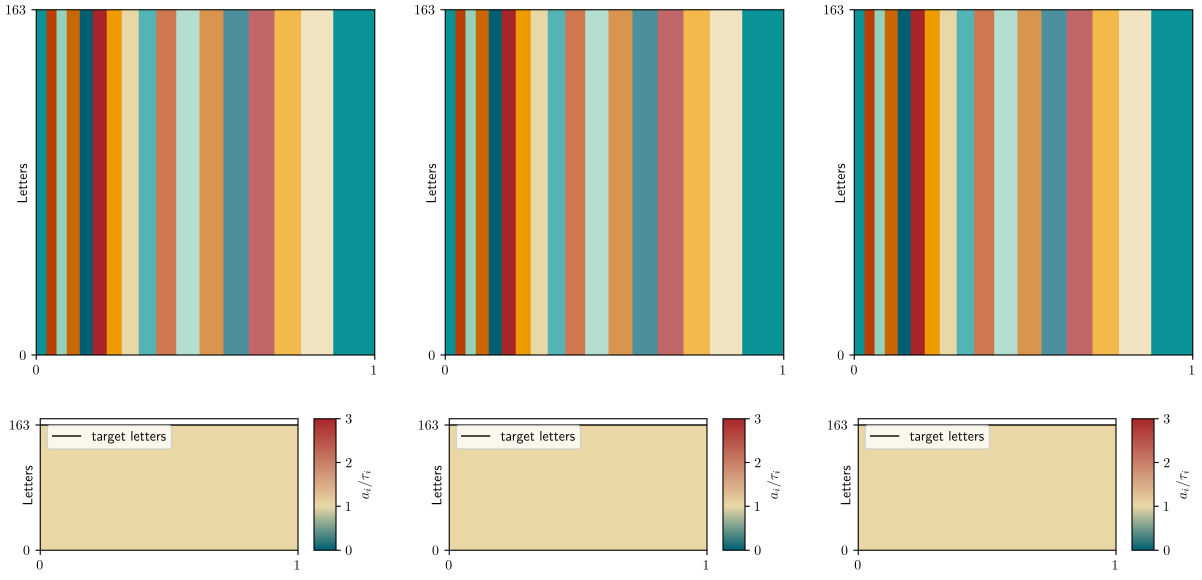
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 2$) (c) Buckets ($t_G = 2$)

Figure 32: Small municipalities of Nordrhein-Westfalen ($\ell_G = 553$)



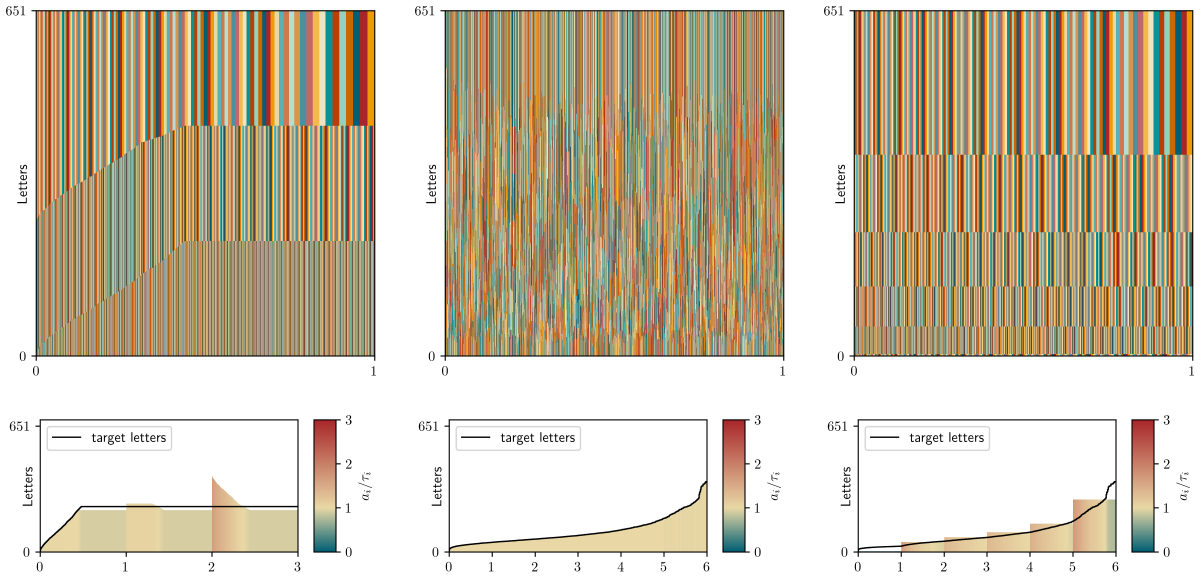
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 33: Large municipalities of Rheinland-Pfalz ($\ell_G = 172$)



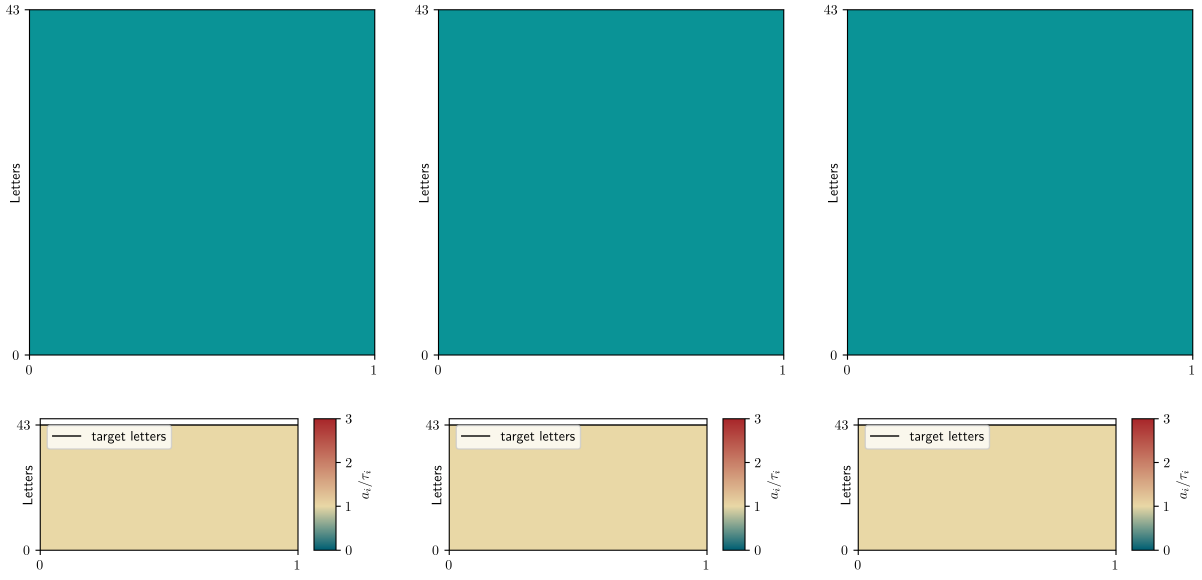
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 34: Medium municipalities of Rheinland-Pfalz ($\ell_G = 163$)



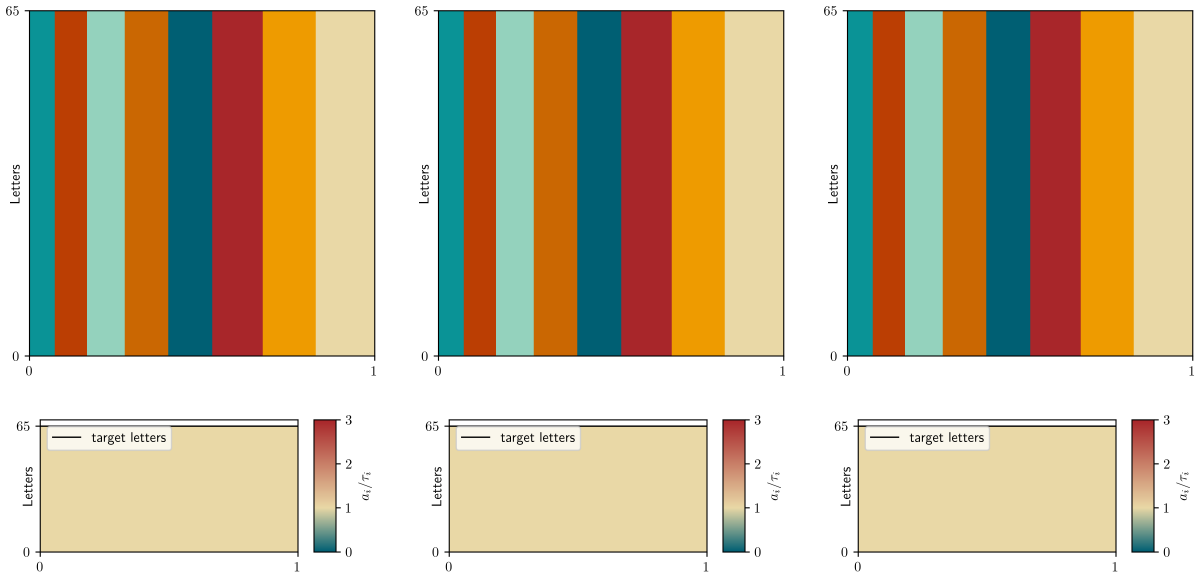
(a) GreedyEqual ($t_G = 3$) (b) ColumnGeneration ($t_G = 6$) (c) Buckets ($t_G = 6$)

Figure 35: Small municipalities of Rheinland-Pfalz ($\ell_G = 651$)



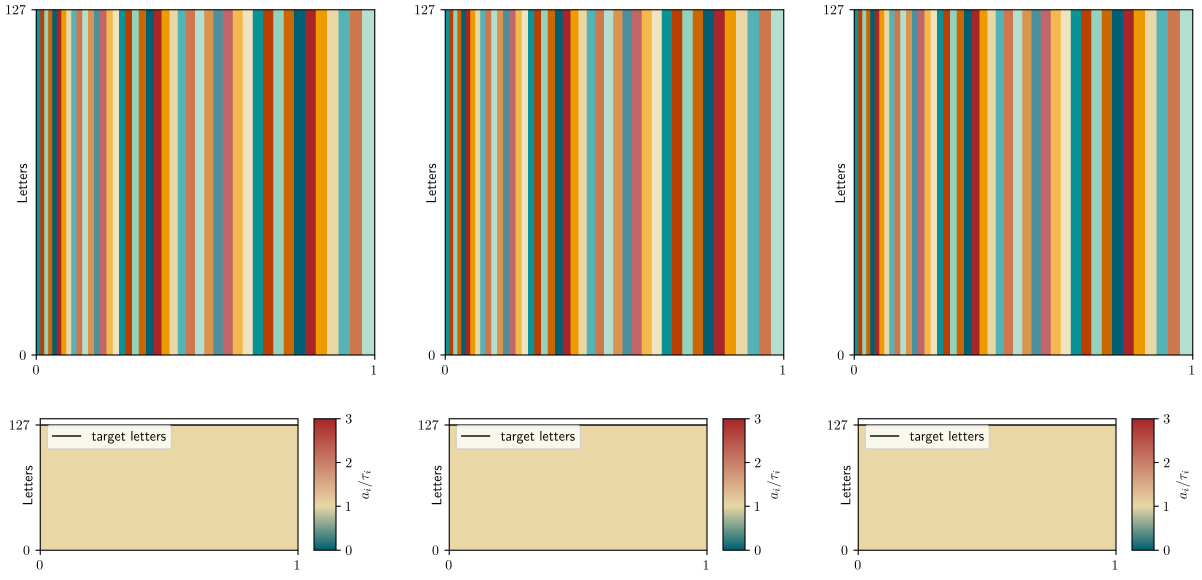
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 36: Large municipalities of Saarland ($\ell_G = 43$)



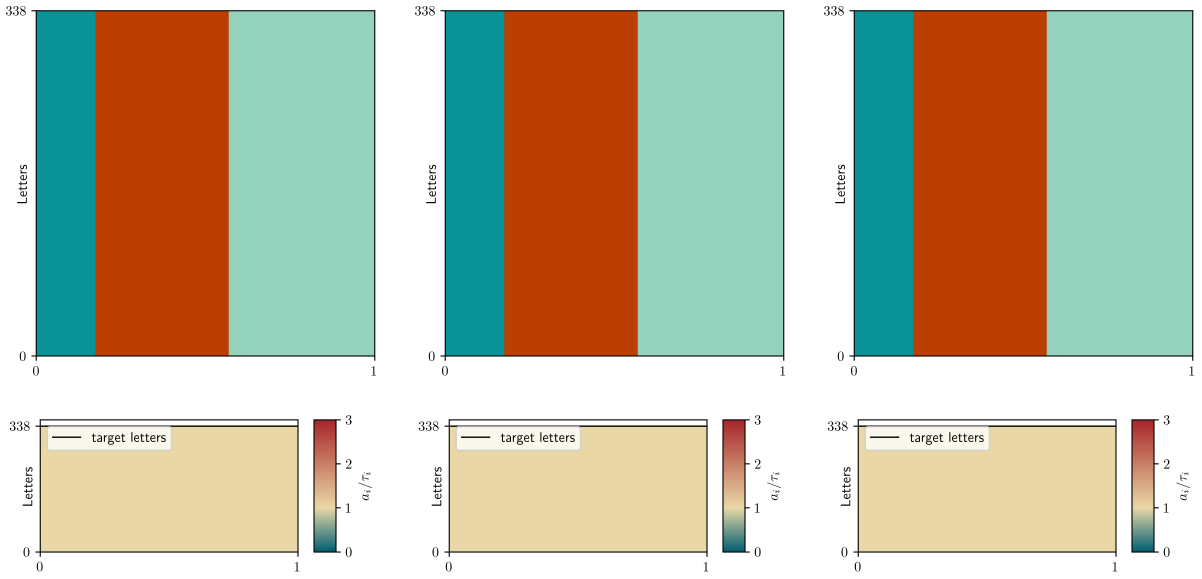
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 37: Medium municipalities of Saarland ($\ell_G = 65$)



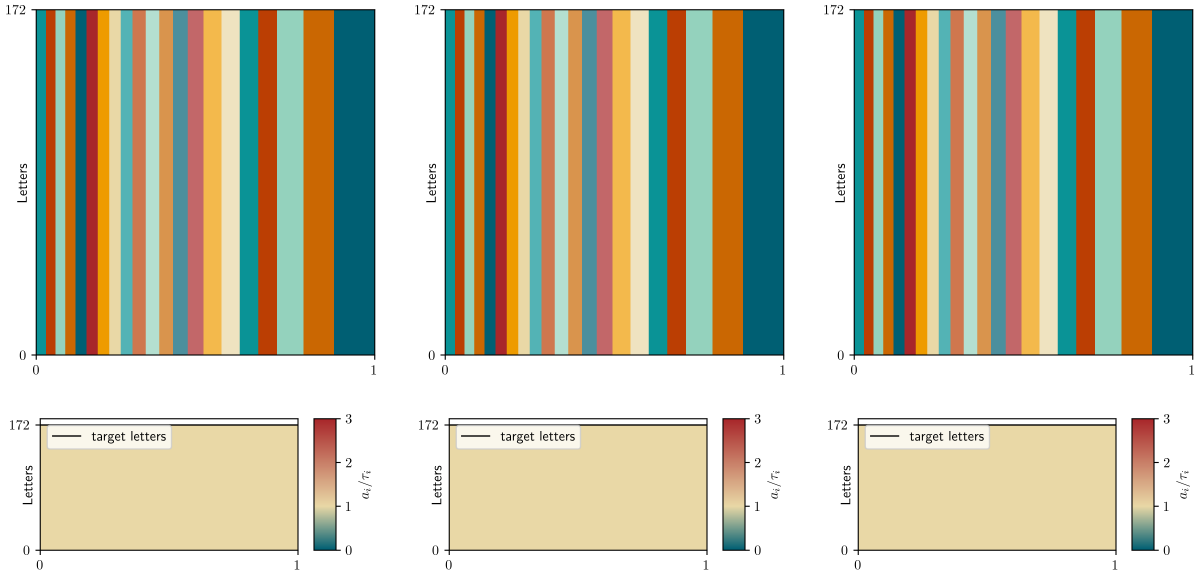
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 38: Small municipalities of Saarland ($\ell_G = 127$)



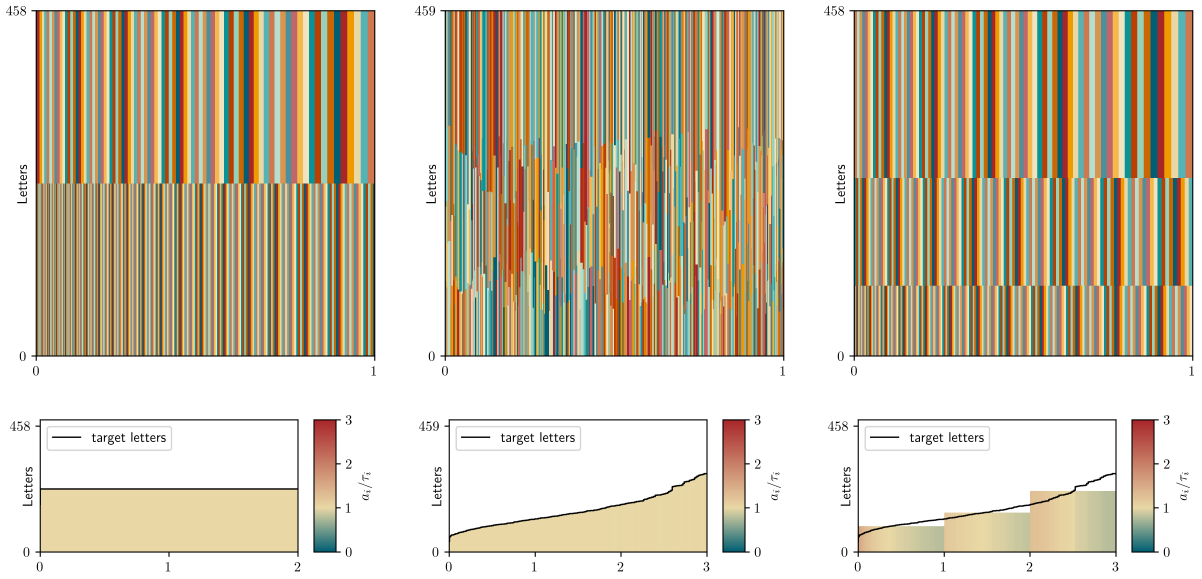
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 39: Large municipalities of Sachsen ($\ell_G = 338$)



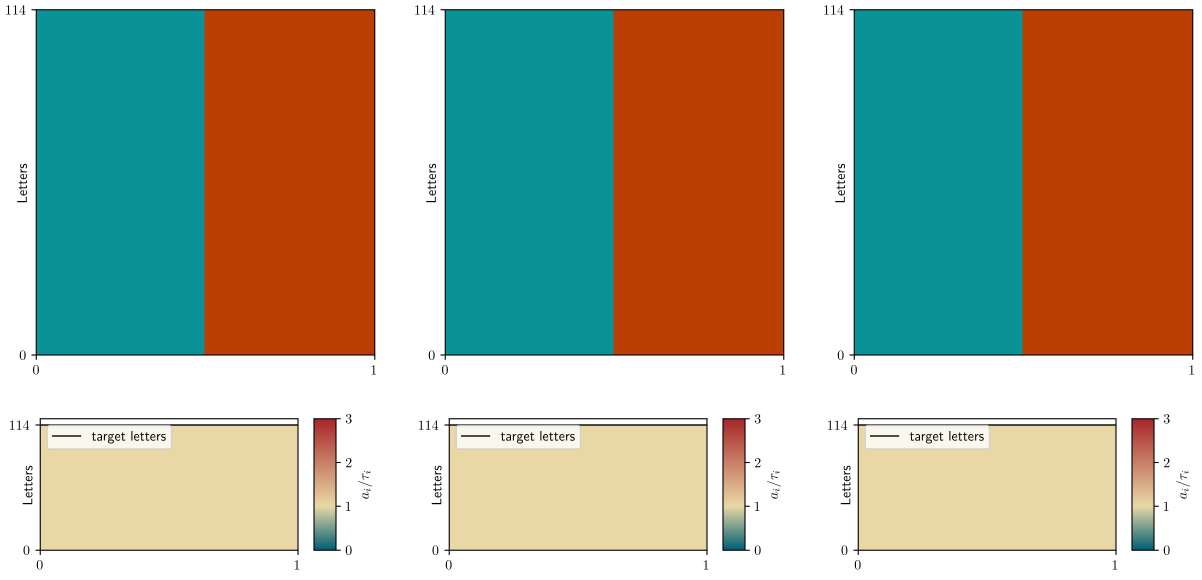
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 40: Medium municipalities of Sachsen ($\ell_G = 172$)



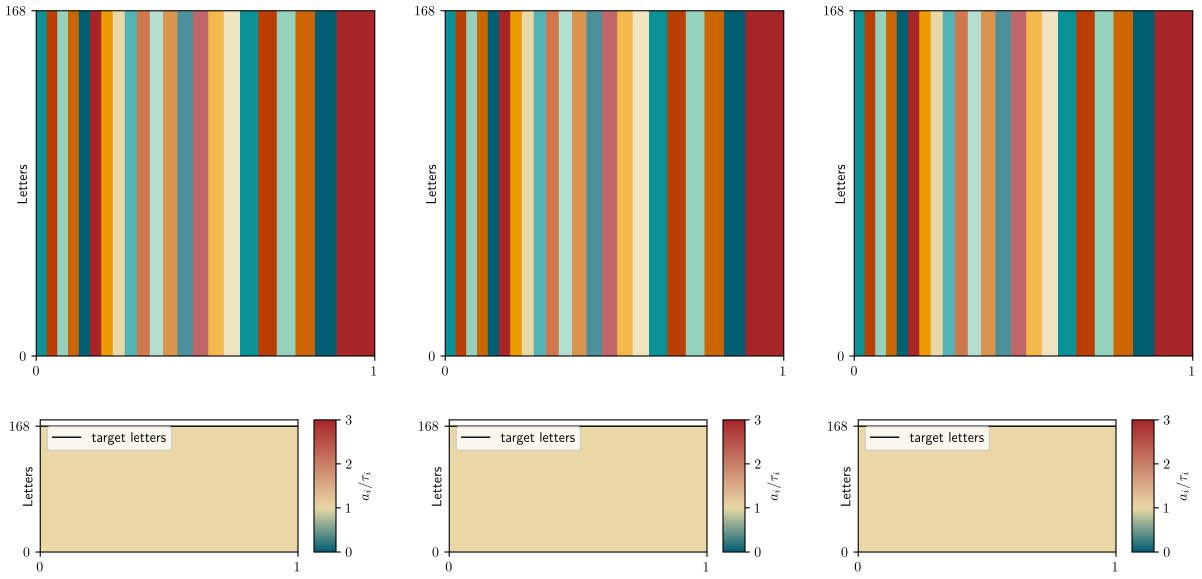
(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 3$) (c) Buckets ($t_G = 3$)

Figure 41: Small municipalities of Sachsen ($\ell_G = 458$)



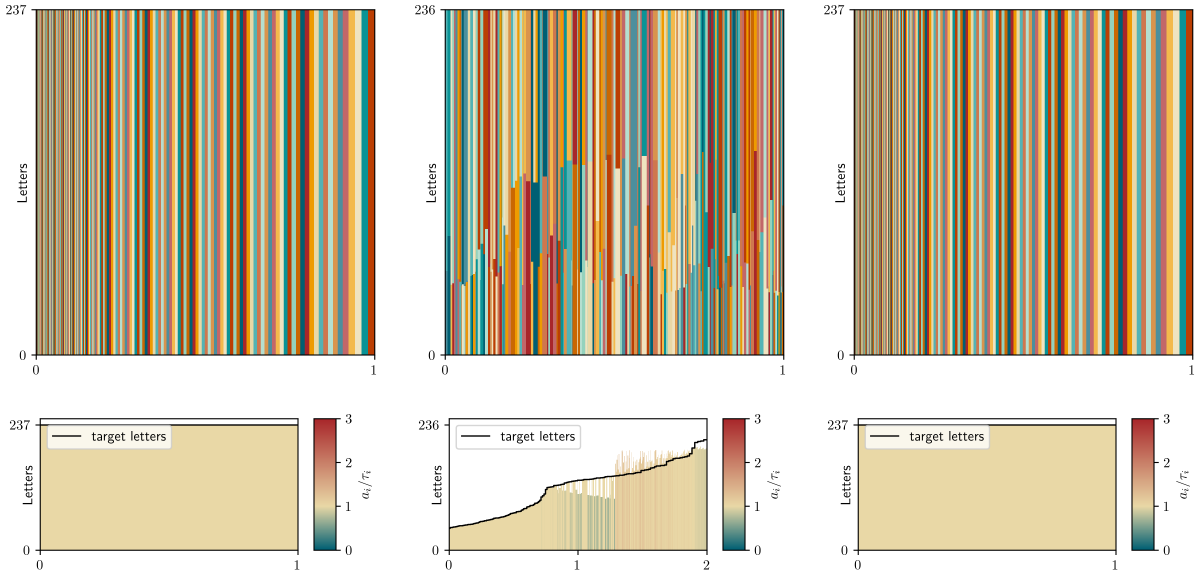
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 42: Large municipalities of Sachsen-Anhalt ($\ell_G = 114$)



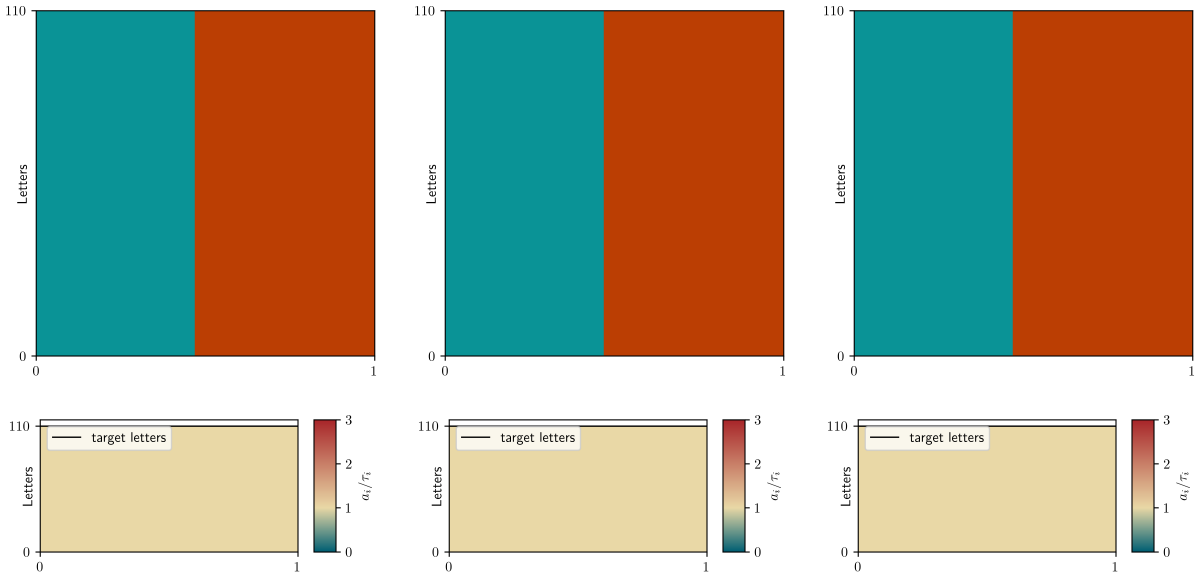
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 43: Medium municipalities of Sachsen-Anhalt ($\ell_G = 168$)



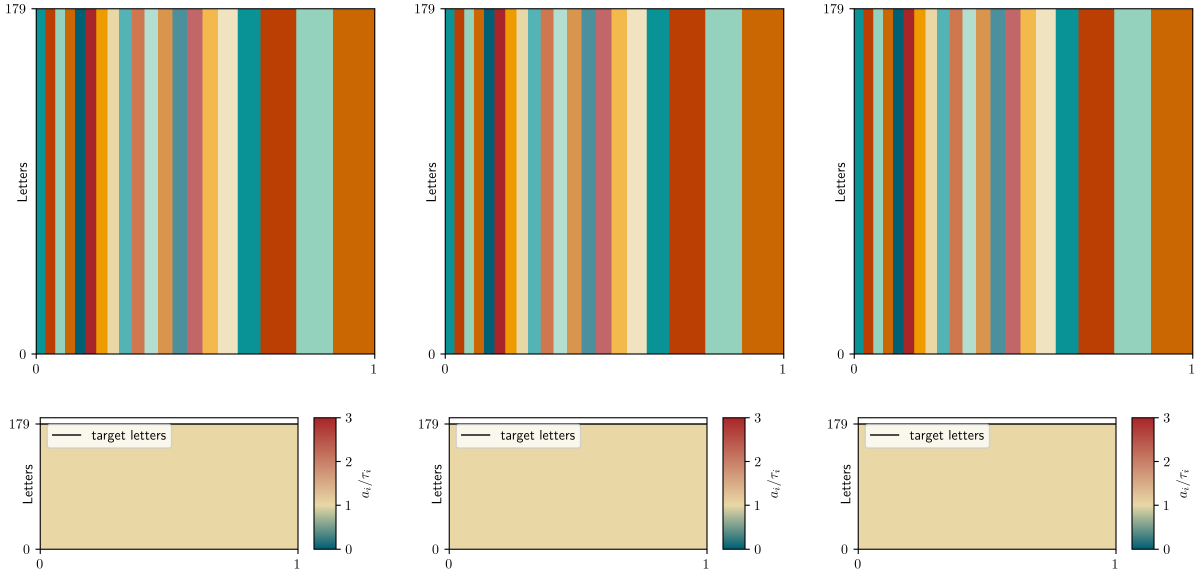
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 2$) (c) Buckets ($t_G = 1$)

Figure 44: Small municipalities of Sachsen-Anhalt ($\ell_G = 237$)



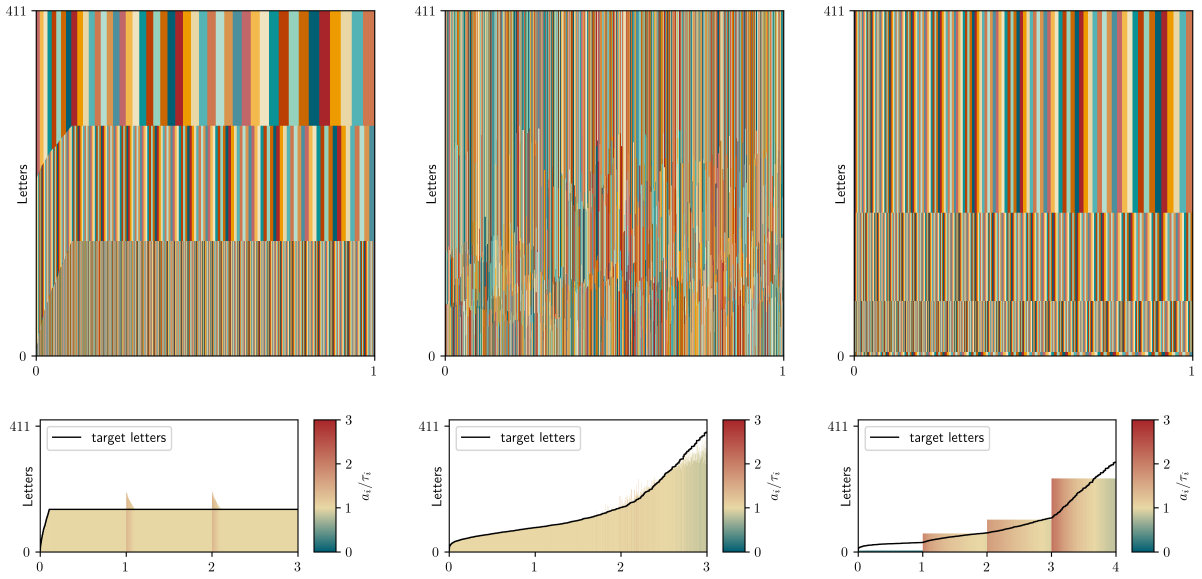
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 45: Large municipalities of Schleswig-Holstein ($\ell_G = 110$)



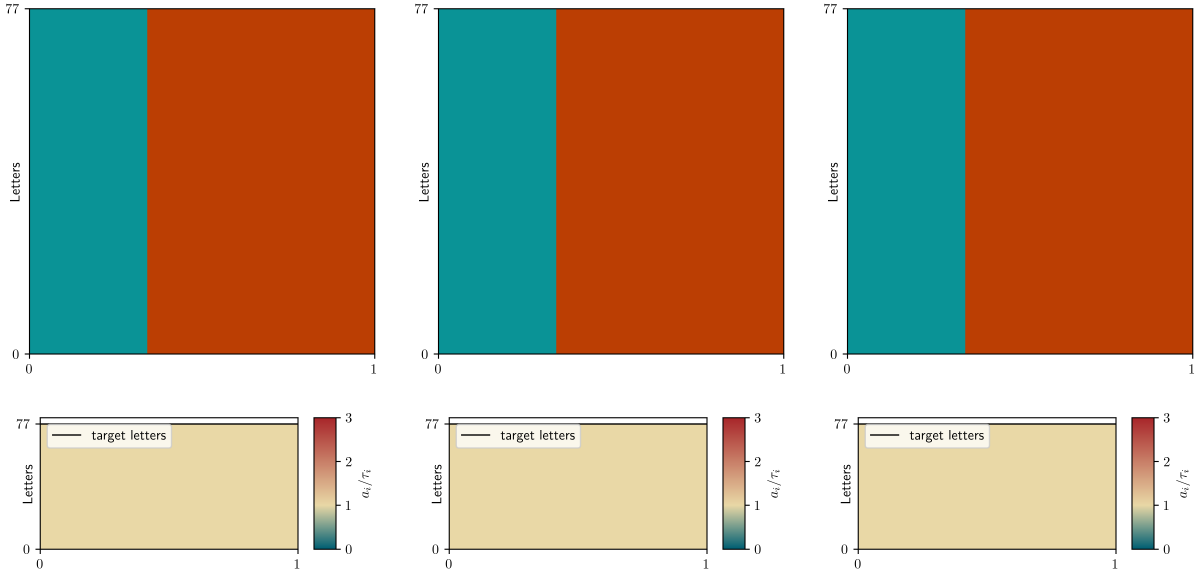
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 46: Medium municipalities of Schleswig-Holstein ($\ell_G = 179$)



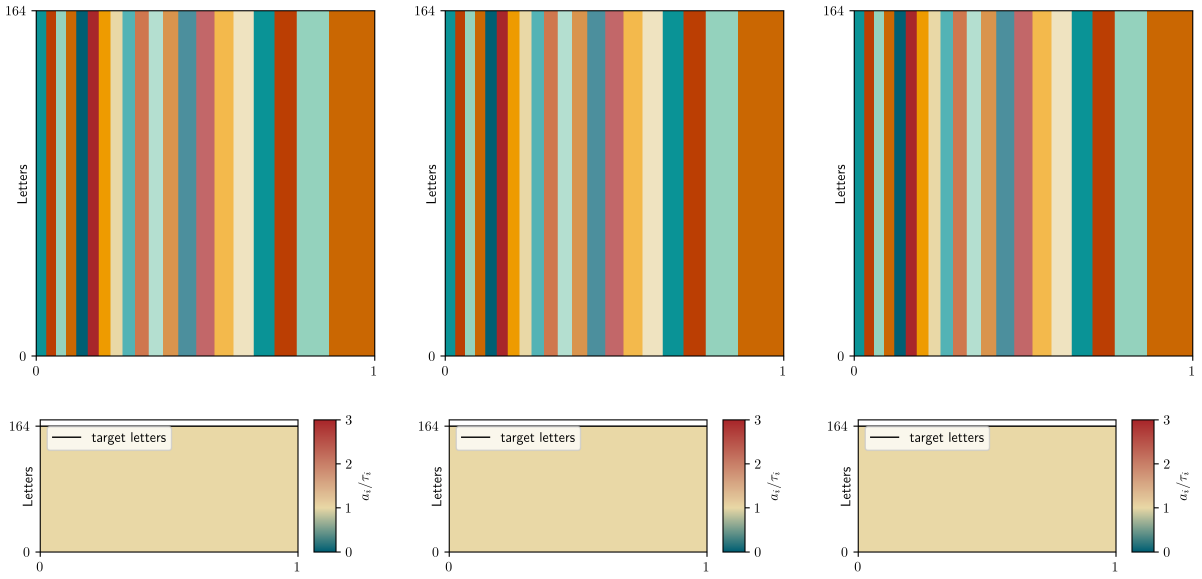
(a) GreedyEqual ($t_G = 3$) (b) ColumnGeneration ($t_G = 3$) (c) Buckets ($t_G = 4$)

Figure 47: Small municipalities of Schleswig-Holstein ($\ell_G = 411$)



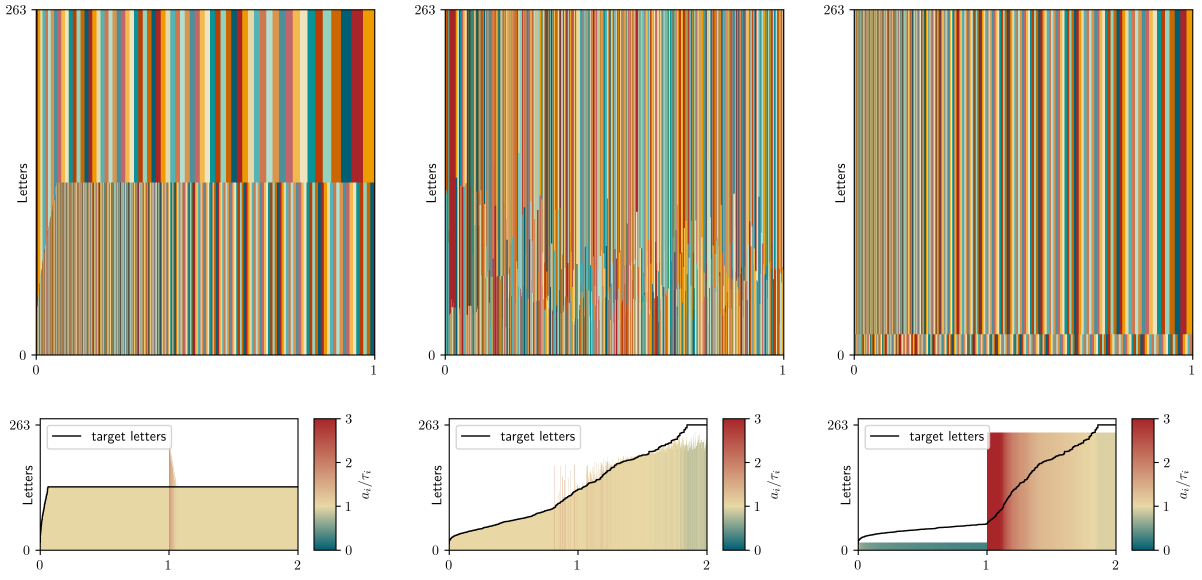
(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 48: Large municipalities of Thüringen ($\ell_G = 77$)



(a) GreedyEqual ($t_G = 1$) (b) ColumnGeneration ($t_G = 1$) (c) Buckets ($t_G = 1$)

Figure 49: Medium municipalities of Thüringen ($\ell_G = 164$)



(a) GreedyEqual ($t_G = 2$) (b) ColumnGeneration ($t_G = 2$) (c) Buckets ($t_G = 2$)

Figure 50: Small municipalities of Thüringen ($\ell_G = 263$)

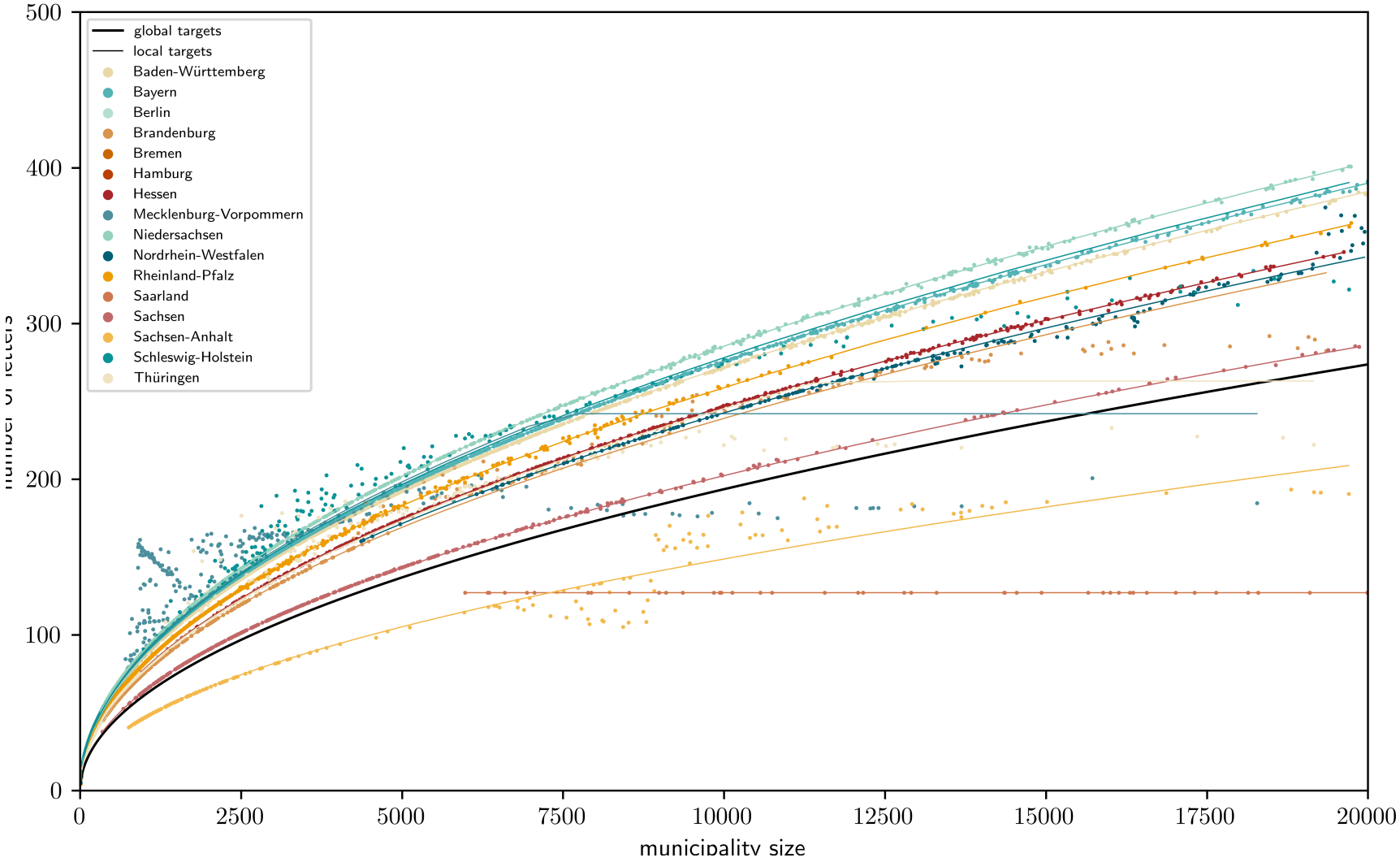


Figure 51: ColumnGeneration for groups with small cities

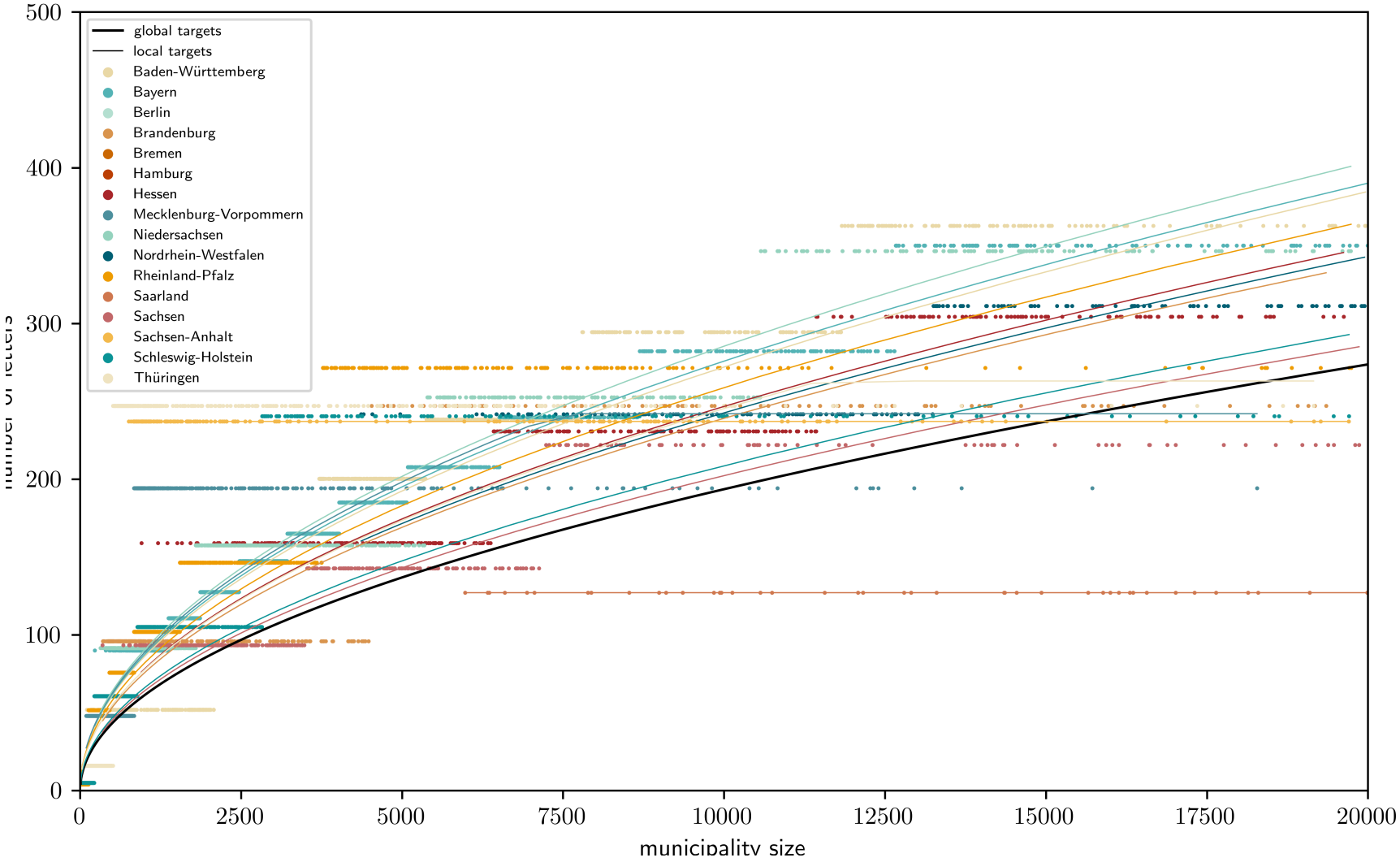


Figure 52: BUCKETS for groups with small cities

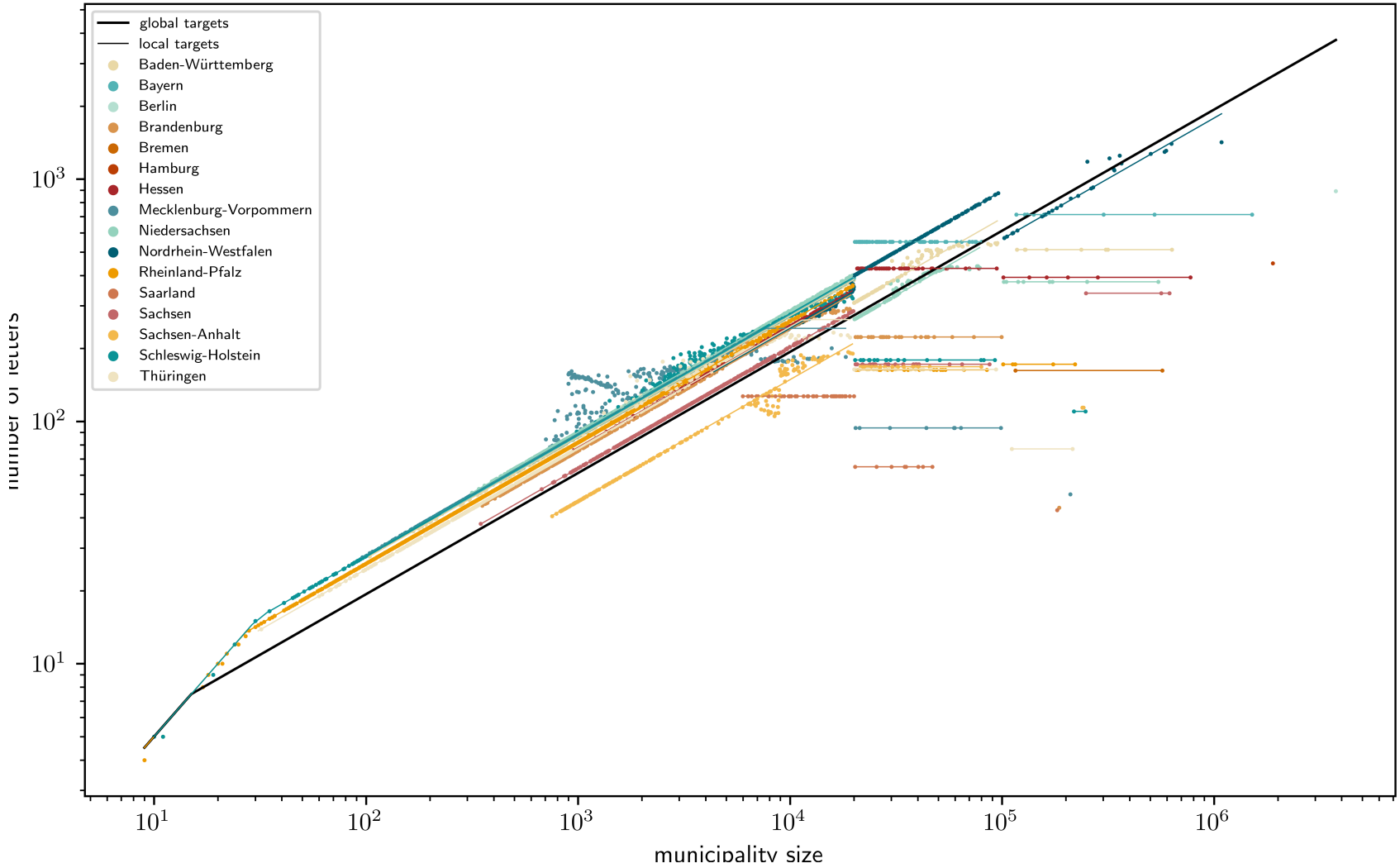


Figure 53: ColumnGeneration for all groups

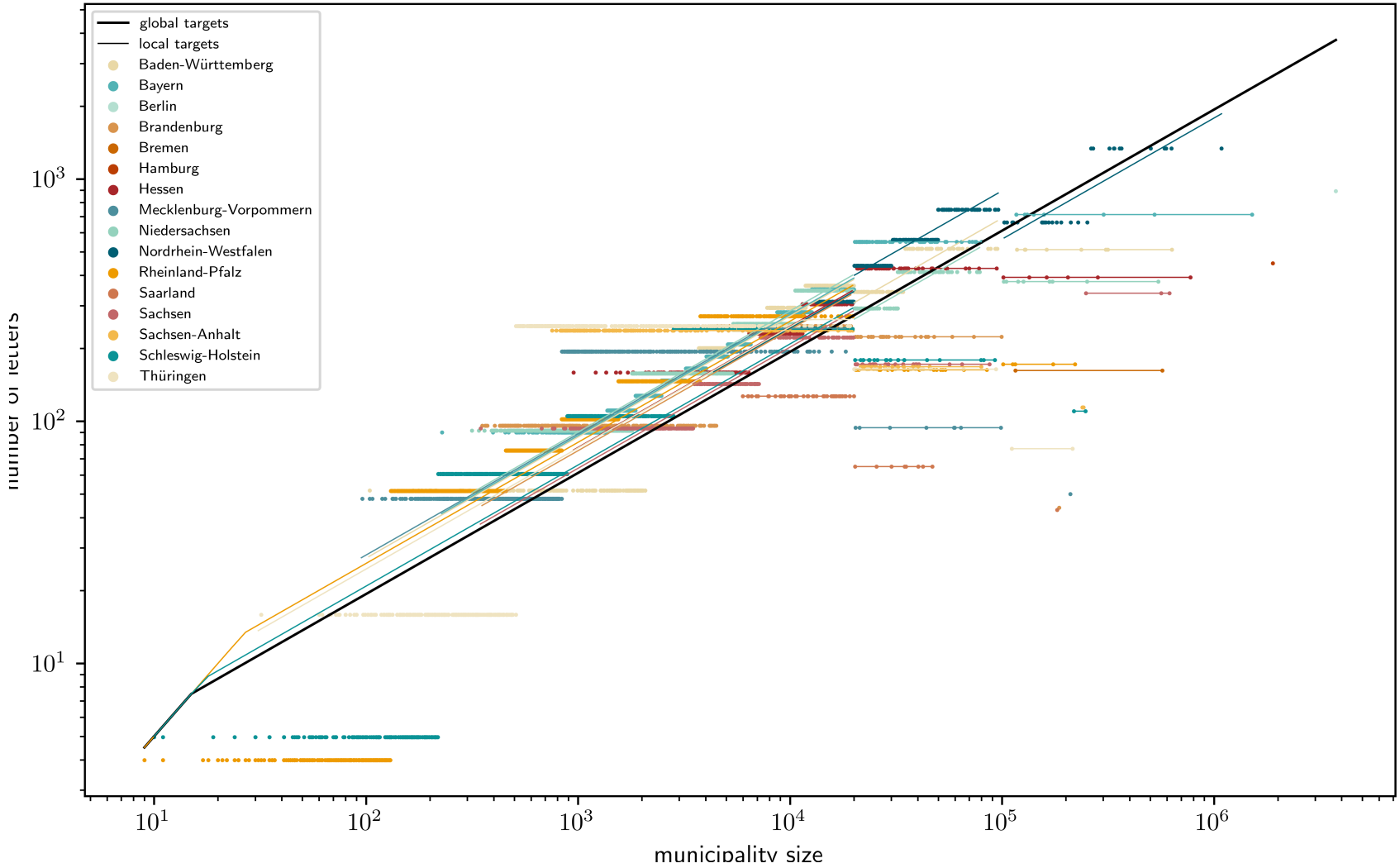
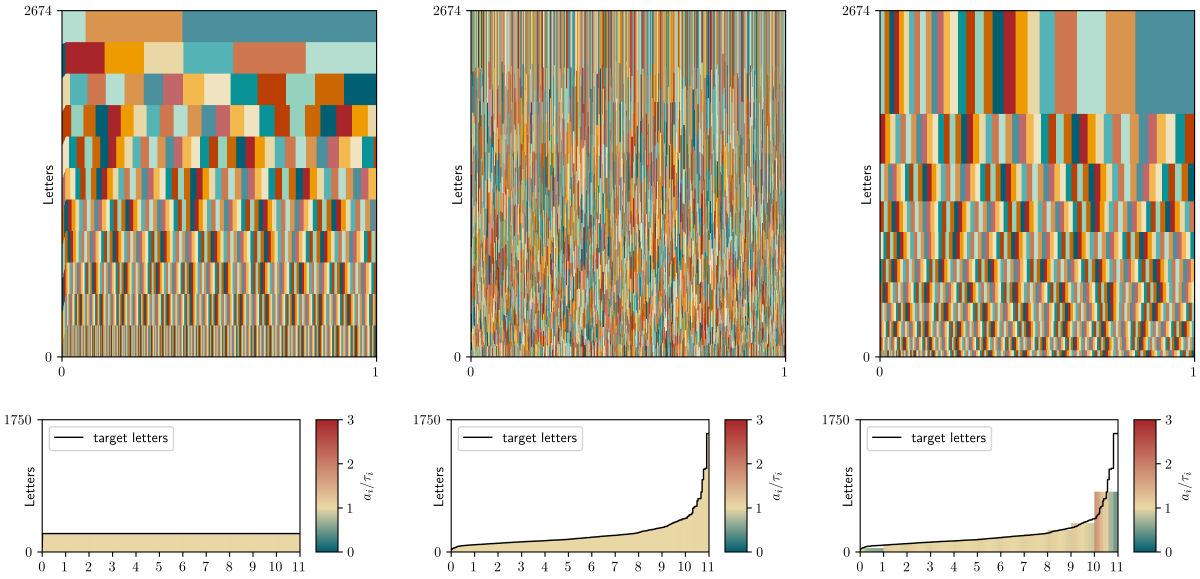


Figure 54: BUCKETS for all groups



(a) GreedyEqual ($t = 11$) (b) ColumnGeneration ($t = 11$) (c) Buckets ($t = 11$)

Figure 55: All cities of Baden-Württemberg ($\ell = 2674$)